From Quantum Fields to Cosmic Structures: Tracing the Early Universe through Correlation Functions

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Abstract

Our current understanding of gravity is limited by the challenge of probing extremely high energy scales, which are far beyond the reach of terrestrial experiments. Theoretical frameworks such as string theory and loop quantum gravity remain largely untested, as building a particle collider capable of accessing these scales would require structures on the order of the solar system. However, the early universe—particularly during the inflationary epoch—naturally operated at such energy scales, offering a unique observational window into high-energy physics.

Inflation is the highest-energy process known to have occurred in the history of the universe, making it a compelling setting to investigate particle production, field interactions, and signatures of new physics. Crucially, these phenomena can leave observable imprints in the cosmic microwave background (CMB), particularly in the form of primordial correlation functions.

This thesis explores how particles produced during inflation—via gravitational effects or interactions—affect cosmological observables such as the power spectrum, bispectrum, and trispectrum. We employ tools from quantum field theory in curved spacetime, including Bogoliubov transformations to understand particle production and the in-in formalism, to compute these correlation functions and interpret their physical implications. Special attention is given to non-Gaussian features, which serve as powerful diagnostics of field content and interactions during inflation.

By treating the early universe as a natural high-energy particle detector, we explore the connection between the theoretical machinery of quantum field theory with the observational data encoded in the CMB. The resulting framework enables us to test inflationary models and search for evidence of new physics beyond the Standard Model—buried in the statistical patterns of the sky.

Contents

Chapter 1

Introduction and Motivation

Who are we? Where did we come from? What's the purpose of life? These timeless questions have haunted ancient philosophers for centuries—usually while they stared into campfires or scribbled on scrolls with dramatic sighs. While answers to some of these remain elusive, today we tackle a slightly more manageable one from the cosmic mystery grab bag: Where did everything come from?

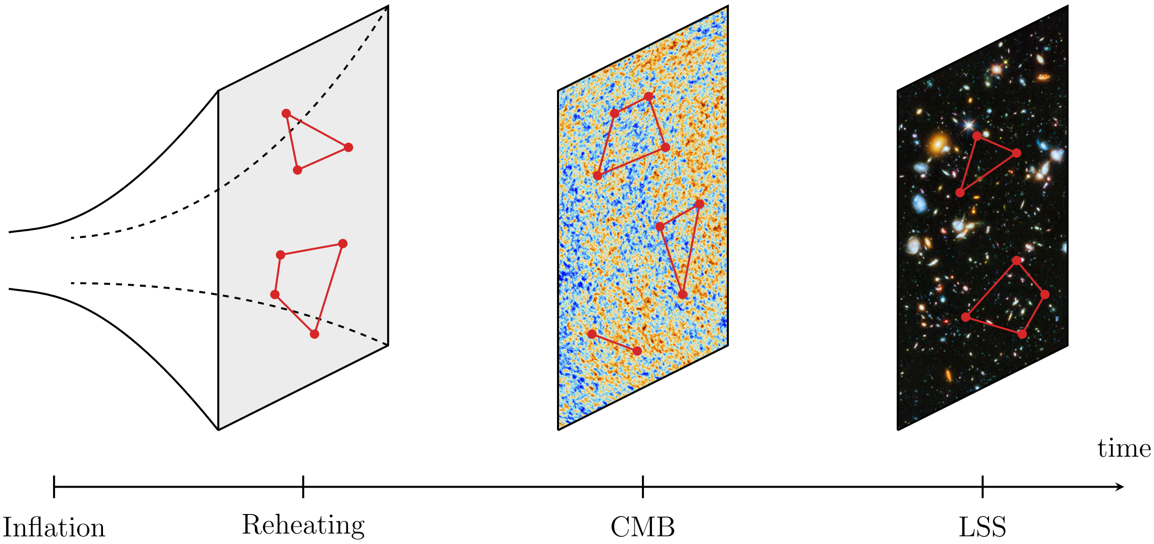


Figure 1.1: Correlations in late-time cosmological observables—like LSS distribution or CMB anisotropies—originate from primordial correlations on the reheating surface, which marks the end of inflation and the start of the hot Big Bang. These correlations carry imprints of inflationary physics. Figure adapted from [1].)

Cosmology is the branch of science that attempts to answer this question, among others. In this report, our focus will remain on inflation and what it could teach us about our universe. The birth of the universe and the subsequent evolution that led to the formation of large-scale structures is one of the most fundamental and profound questions in cosmology. Our current understanding of Large Scale structure formation is based on cosmological perturbation theory. It gives us a set of differential equation which govern the linear time evolution of growth of structure. However, as for any system which is evolved by a set of differential equation from the past to infinite future, one needs the set of initial conditions. The initial conditions set by quantum fluctuations which took place during inflation and they obey gaussian probability distribution.

It is not the only problem that Inflation solves, but the motivation for considering inflation arises naturally when we consider the framework that Hot Big Bang Cosmology provides. It had many problems, such as[2]

- Horizon Problem: Why does the Cosmic Microwave Background Radiation resemble the Black Body radiation emitted by object at thermal equilibrium at $T = 2.7K$, if the distant patches in the sky were not in causal contact. Which mechanism did the universe use to establish thermal equilibrium?

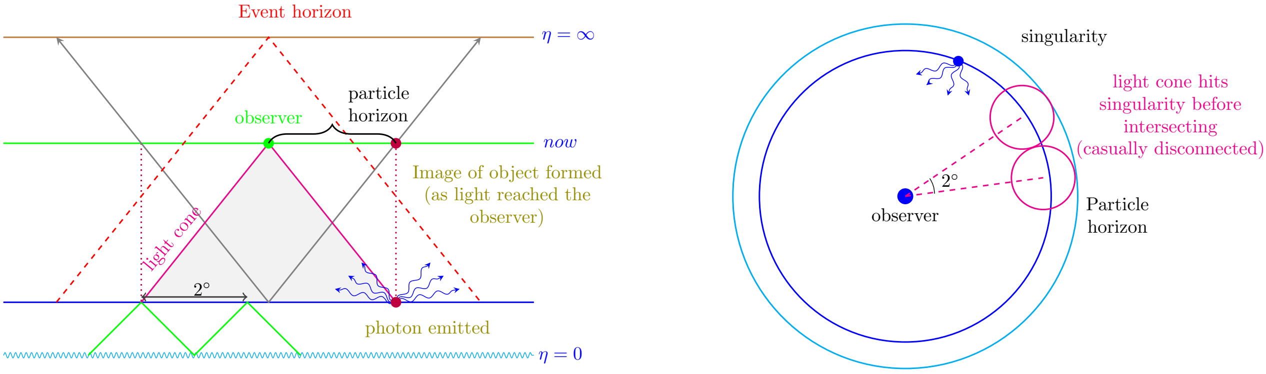


Figure 1.2: Two views of the horizon problem. **Left:** Conformal diagram showing light from two regions of the sky (separated by $\sim 2°$) reaching the observer today. Their past light cones hit the initial singularity before intersecting, implying no causal contact. **Right:** Spatial slice at constant time illustrating the same: regions lie outside each other's particle horizons. Yet, they exhibit identical CMB temperatures.

- Flatness problem: How the universe could have such a precise density of matter and energy to be so close to flat. It could be thought of as fine tuning problem in cosmology.
- Monopole problem: Exotic unification models like GUT predict the existence of magnetic monopole and yet none has been observed so far.

These problems among others can be solved by assuming an exponentially expanding phase preceding the Hot big bang model. Inflation makes it possible for the entire observable universe to have had been in causal contact in the past so that thermal equilibrium could be established which is what the Horizon problem is all about. The rapid expansion also flattens out any spatial curvature as well as dilutes the abundance of unwanted relic such as magnetic monopole.

Another interesting idea of interest in cosmology is that of Gravitational Particle Production (GPP), which suggests a possible mechanism for producing matter in the Early Universe. GPP is the creation of particles from vacuum in the expanding universe (similar to how Black Holes radiate) entirely due to their gravitational interaction. The idea is somewhat similar to production of electron and positron in the presence of strong electric field[3] with gravitational field replaced by strong electric field.

The vacua of quantum fields in the early universe are among the most fascinating states to study, primarily because the definition of the vacuum — understood as a state with no particles — evolves over time due to gravitational dynamics. This idea is analogous to a forced harmonic oscillator in quantum mechanics. If an impulse is applied to a harmonic oscillator in its ground state, it becomes excited to higher energy states, exhibiting oscillatory behavior. In the context of inflationary cosmology, this implies that a state initially devoid of particles becomes populated with them due to gravitational effects. When quantum fields are interpreted as harmonic oscillators in momentum space, their behavior in curved spacetime resembles that of a forced harmonic oscillator. This is the idea behind gravitational production of particles during inflation. In the adiabatic limit, i.e. slowly expanding universe scenario, we observe no particle production[4].

One of the most beautiful consequences of considering Inflation is that it is one of the highest-energy phenomena ever to take place in the universe. The highest energy scale that one can reach at the Large Hadron Collider is in the few TeV range, whereas inflation took place at the energy scale of 10^{14} GeV. Inflation not only solves a few problems but also introduces many more. The most fundamental being the observational effects that one can measure.

The scientific method demands that we experimentally recreate the phenomenon over and over again to observe the same effect and ensure that the process being studied is not the result of statistical fluctuations. However, the birth of the universe was a phenomenon that cannot be recreated in a lab. Therefore, it is natural to ask how we can infer the high-energy processes or GPP that took place during inflation. Because, inflation was very high energy phenomenon, which means that during this phase, particles lighter than inflationary expansion scale could have been produced. However, since, we have no direct way of accessing the inflationary era. How do we recognize the presence of these new particles during inflation from CMB? More importantly, why isn't the structures in the universe randomly distributed? One such resolution lies in studying the cosmological correlation functions and use them to infer particle production signatures and other events which took place during inflation from the CMB. Correlation functions in field theories are expectation values of field operator between vacuum states.

$$\langle OOX \rangle = \langle 0 | OOX | 0 \rangle$$

These correlations are like the fossil record of the universe which could help us understand the origin and evolution of cosmological perturbation.

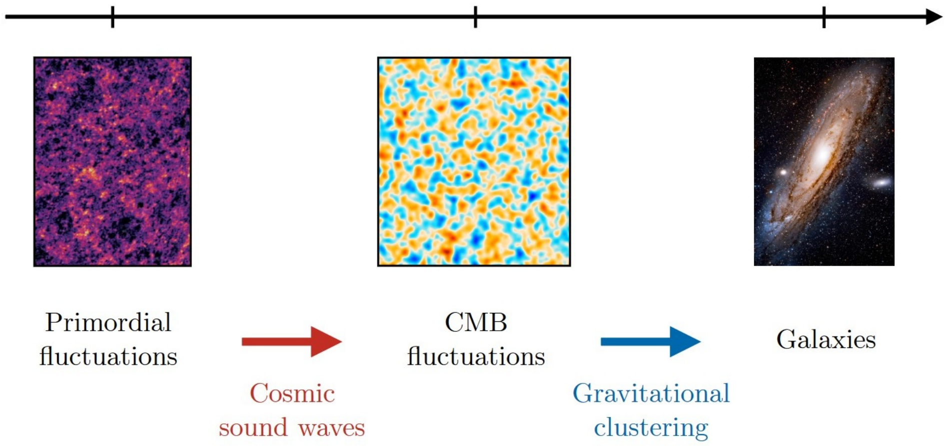


Figure 1.3: From quantum seeds to cosmic structure: Primordial fluctuations, seeded by quantum fluctuations during inflation, were amplified by gravity and shaped by plasma physics into the anisotropies seen in the CMB. Over time, gravitational clustering transformed these fluctuations into the large-scale structure we observe today, including galaxies and galaxy clusters.

It is clear that studying correlation functions and CMB holds the key to understanding much of the early universe. We quickly motivate why one should consider studying this from the perspective of CMB Physics. The farthest we can look back in time is the surface of last scattering, where the CMB was released. The CMB is not isotropic but has small anisotropies which varies with position. These anisotropies should not have existed if the early universe had been in perfect thermal equilibrium; instead, they reflect initial quantum fluctuations in density, which were later amplified by gravitational instability. These fluctuations correspond to variations in matter density at the time when the first atoms formed. Over time, collisionless dark matter began to accumulate, creating gravitational potential wells into which baryonic matter fell, eventually leading to the formation of large-scale structures. This suggests that a natural place to start investigating the origin of large-scale correlations is with the correlations sourced by quantum fluctuations during inflation. These quantum fluctuation set the initial condition for each perturbation mode during radiation dominated era whose evolution lead to the large scale structure formation that we observe today.

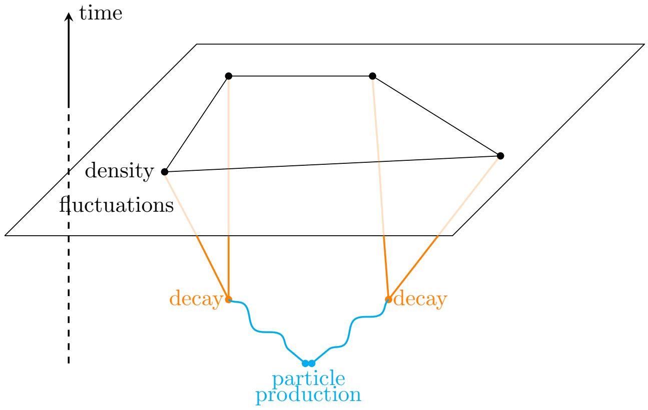


Figure 1.4: A schematic diagram of how particle production during inflation lead to the development of density fluctuation in the early universe.

1.1 Goals and Structure of This Thesis

In this work, we explore how the correlation functions generated during inflation can be used to study particle production, field interactions, and possible signals of new physics. We begin by reviewing the statistical framework of cosmological fluctuations and the in-in formalism for calculating correlation functions in time-dependent spacetimes. We then study quantum field theory in curved backgrounds, focusing on the observer-dependence of vacuum states and the mechanics of particle production.

Subsequent chapters investigate how these phenomena manifest in the CMB, with emphasis on non-Gaussian features that encode rich information about the inflationary era. The tools developed here bridge the gap between high-energy particle physics and observational cosmology, offering a way to test fundamental physics through the lens of the sky.

Ultimately, this thesis aims to explore the growing body of work that treats the early universe as a cosmic detector—an arena where the deepest laws of nature may have left behind observable signatures.

Chapter 2

Statistics of Fluctuations

Inflationary cosmology provides a natural mechanism for the generation of primordial density perturbations from quantum fluctuations of fields in a rapidly expanding background. These fluctuations serve as the initial seeds for the cosmic microwave background (CMB) anisotropies and the large-scale structure we observe today. A central aspect of modern cosmology is to understand the statistical properties of these fluctuations and extract information about the physics of the early universe from their imprints.

In the simplest models, these primordial fluctuations are predicted to be nearly Gaussian and scale-invariant. This prediction arises from the assumption that quantum fields begin in the Bunch-Davies vacuum—a vacuum state which coincides with the Minkowski vacuum at short distances and smoothly evolves with the background spacetime. Each Fourier mode of the field behaves as an independent quantum harmonic oscillator, and the ground state wavefunction for such a system is a Gaussian. Consequently, the probability distribution for the primordial fluctuations is also Gaussian to leading order.

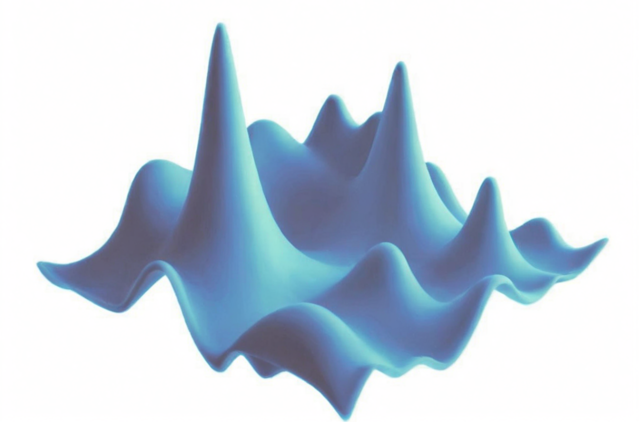


Figure 2.1: Visualization of a 2D Gaussian random field representing fluctuations in an early-universe-like setting. Each bump corresponds to a local deviation from the average — a "quantum wrinkle" in the primordial fabric. While the bumps appear random, their underlying pattern is governed by statistical correlations, which we study through tools like the power spectrum and correlation functions.

In this chapter, we aim to understand where this Gaussianity comes from. Specifically, we ask: *What is the probability distribution for a single perturbation mode during inflation?* To answer this, we work in the Schrödinger picture and treat each mode as a time-dependent quantum harmonic oscillator.

$$H = \frac{p^2}{2m} + V(x)$$

In Quantum Mechanics, we start with a wave function, which is defined as

$$\psi(x) \equiv \langle x|\psi\rangle \qquad \text{with} \qquad \hat{x}\,|x\rangle = x\,|x\rangle$$

and, we have (setting $\hbar = 1$)

$$[\hat{x}, \hat{p}] = i$$

The schrodinger equation (with flat metric) becomes:

$$i\frac{\partial}{\partial t}\psi = -\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$$

Similarly, in quantum field theory, the Hamiltonian for a scalar field is given as:

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}\omega^2\phi^2$$

We begin by considering the eigenstate of the quantum field operator. $\hat{\Phi}$:

$$\hat{\Phi}(t, x)\,|\phi\rangle = \phi(t, x)\,|\phi\rangle \tag{2.1}$$

where $\phi(t,x)$ is the spatial profile of the field at a given time t. We have

$$\psi[\phi(t,x)] \equiv \langle\phi|\psi\rangle \qquad \text{with} \qquad \hat{\Phi}(t,x)\,|\phi\rangle = \phi(t,x)\,|\phi\rangle$$

and,

$$[\hat{\Phi}(x), \hat{\Pi}(y)] = i\delta(x-y)$$

The schrodinger equation then becomes:[5]

$$i\frac{\partial}{\partial\eta}\psi[\phi(t,x)] = \left(-\frac{1}{2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2}\omega^2\phi\right)\psi[\phi(t,x)]$$

This looks like the Schroedinger equation for harmonic oscillator (with $m = 1$) and thus we know that the wavefunction for ground state is gaussian i.e.

$$\langle\phi|0\rangle = Ce^{-\frac{\omega}{2}\phi(t,x)^2}e^{-iE_0\eta}$$

As long as the primordial density fluctuation has linear dependence over the above quantum field, it should obey a Gaussian distribution. There are other ways to detect whether the distribution is Gaussian or not. By measuring odd-order moments of the distribution function. The bispectrum is the first non-zero moment observed in non-Gaussian distribution, in the sense that if the bispectrum is non-zero, the distribution surely is non-Gaussian. Note that the converse is not necessarily true. Hence, if we measure a non-zero bispectrum, we can be sure that the distribution is non-Gaussian and there are some **interaction** or **particle production** taking place. Measurements of higher-order correlation functions (or non-Gaussianity) are the analog of measuring collisions in particle physics. This is what inspires the study of these non-Gaussianity in the CMB correlation function, to be named "cosmological collider physics"[6].

We can see the same from path integral approach as well. The wavefunction at any given time (which we can set to $t_* = 0$ without loss of generality) can be formally computed by the path integral formalism:

$$\Psi[\phi] = \int \mathcal{D}\Phi\, e^{iS[\Phi]} \approx e^{iS[\Phi_{\rm cl}]}$$

with

$$\Phi(0) = \phi \qquad \Phi(-\infty^+) = 0$$

Consider a scalar field, whose action is

$$S[\Phi] = \int dt\left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2\right),$$

where Φ is the deviation from equilibrium and ω is the constant frequency of the oscillator. On-shell, the action can equivalently be written as a pure boundary term

$$S[\Phi_{\rm cl}] = \int_{t_i}^{t^*} dt\left[\frac{1}{2}\partial_t(\dot{\Phi}_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl})\right] = \frac{1}{2}\dot{\Phi}_{\rm cl}\Phi_{\rm cl}\Big|_{t=t^*},$$

where we have integrated by parts and used the fact that the classical solution satisfies the equation of motion $\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl} = 0$. To determine the classical solution $\Phi_{\rm cl}(t)$ we have to specify two boundary conditions: We require:

1. the late-time value of the oscillator position is $\Phi_{\rm cl}(t_* = 0) = \phi$, and
2. the early-time limit is the minimum-energy solution $\Phi_{\rm cl}(t) \sim e^{i\omega t}$.

The unique solution satisfying these boundary conditions is $\Phi_{\rm cl}(t) = \phi e^{i\omega t}$. Substituting this into above, we get

$$S = \frac{i\omega}{2}\phi^2 \quad\Longrightarrow\quad \exp(iS) = \exp\left(-\frac{\omega}{2}\phi^2\right) \quad\Longrightarrow\quad |\Psi[\phi]|^2 = e^{-\omega\phi^2}.$$

We see the familiar fact that the ground state wavefunction of the simple harmonic oscillator is a Gaussian. The width of this Gaussian determines the size of the zero-point fluctuations of the oscillator:

$$\langle\phi^2\rangle = \frac{1}{2\omega}.$$

We conclude that in the absence of interaction, Inflation to a large extend predicts gaussianity. Therefore, any deviation from gaussianity has the potential to teach us much more about the early universe and inflation that we cannot ignore it. Therefore, it is important for us to understand how to calculate the bispectrum (three point function), trispectrum (four point function), and other measures of non-gaussianity.

2.1 Cosmological Correlation Function

The main object for studying the statistics of fluctuation is the correlation function. Power spectrum, bispectrum or trispectrum are some of the most used correlation function in the study of statistics of the early universe quantum fluctuation. There are mainly three ways to evaluate the correlation function in quantum field theory relevant for cosmological study. They are as follows:

- In-In Formalism
$$\langle \mathcal{O}(t)\rangle = \langle\Omega|\, \overline{T} e^{i\int_{t_i}^{t} dt' H_{\rm int}(t')} \mathcal{O}_I(t) T e^{-i\int_{t_i}^{t} dt' H_{\rm int}(t')} \,|\Omega\rangle\,,$$
where $\overline{T}$ denotes anti time-ordering.
- Wavefunction Approach
- Bootstrap

However, we will limit ourselves to In-In formalism, but in the next chapter we will explore few aspects of Bootstrap philosophy.

2.1.1 In-In Formalism

One of the starting assumption of S matrix formalism is the requirement of adiabatic switching[7]. We need the interactions to be turned "on" and turned "off" adiabatically. When the interaction is turned "on" and "off" adiabatically, the initial and final states are eigenstate of free Hamiltonian. If we now, consider a non-equilibrium process such as inflationary cosmology, where the interaction never ceases and particles are still evolving. The dynamical coupling due to gravity becomes important around the Hubble radius and thereafter, such that there is no "out" state in this case. Thus, Out of equilibrium[1], we can neither apply the adiabatic theorem nor specify the final asymptotically free state and thus the standard S-matrix formalism breaks down. Even if, we could turn the interaction off in asymptotic future, if the $|0\rangle_{\rm out}$ and $|0\rangle_{\rm in}$ are not same upto a phase, as is the case in Hawking radiation. The in-out formalism will not work in such case.

Therefore, we have to look for alternate formalism which can deal with this problem. In the presence of non-equilibrium perturbation, the asymptotic initial and final states do not even belong to the same hilbert space. The idea of in-in formalism is to avoid any reference of initial and final state. To avoid any reference to the final state in the asymptotic future, the trick is to simply rewind the time evolution back to the asymptotic past and hence the name, "in-in" formalism.

Let's start with defining an **equal time** *in-in* correlator as the expectation value of some operator $\mathcal{O}$ with respect to some state $|\Omega\rangle$ in Heisenberg Picture:

$$\langle \mathcal{O}\rangle = \langle\Omega|\, \mathcal{O}\, |\Omega\rangle$$

Since we assumed $\mathcal{O}(t)$ to be product of equal time operators acting at different point in space, it is important to mention the consequence of such choice. Time ordering of $\mathcal{O}$ is therefore irrelevant. We will assume $\mathcal{O}(t)$ to be Hermitian as well, therefore:

$$\langle\Omega|\, \mathcal{O}\, |\Omega\rangle^* = \langle\Omega|\, \mathcal{O}^\dagger\, |\Omega\rangle = \langle\Omega|\, \mathcal{O}\, |\Omega\rangle \in \mathbb{R}.$$

We will use the interaction picture for evaluating the correlation functions, therefore we will utilize the following property[8]:

$$\mathcal{O}_H(t) = U_I^\dagger(t,0)\mathcal{O}_I(t)U_I(t,0) \qquad \text{and} \qquad |\Omega\rangle_H = U_I(0,t)\,|\Omega(t)\rangle_I$$

with the assumption that both pictures coincide at $t = 0$. All that remains now is to define the $|\Omega\rangle$. Since we will only be interested in the case where $|\Omega\rangle$ is the vacuum of interacting theory which in the far past asymptotes the free theory vacuum $|0\rangle$, defined by $a_k|0\rangle = 0$ (but $a_k|\Omega\rangle \neq 0$).

$$\lim_{t\to-\infty} |\Omega\rangle = |0\rangle$$

We assume that, $|\Omega\rangle$ minimizes both $H_{\rm full}$ as well as $H_{\rm int}$, therefore we can use following reasoning to define the above.

$$e^{-iH_{\rm int}(t-t_i)}\,|\Omega\rangle = \sum_n e^{-iH_{\rm int}(t-t_i)}\,|n\rangle\langle n|\Omega\rangle$$

[1] by equilibrium we mean, the final state vector is not changing with time

$$= e^{-iE_0(t-t_i)} |0\rangle \langle 0|\Omega\rangle + \sum_{n\neq 0} e^{-iE_n(t-t_i)} |n\rangle \langle n|\Omega\rangle$$

under $t_i \to t_i(1-i\epsilon)$

$$= e^{-iE_0(t-t_i)} e^{\epsilon E_0 t_i} |0\rangle \langle 0|\Omega\rangle + \sum_{n\neq 0} e^{-iE_n(t-t_i)} e^{\epsilon E_n t_i} |n\rangle \langle n|\Omega\rangle$$

In the limit $t_i \to -\infty$, we have:

$$\lim_{t_i\to-\infty} e^{-iH_{\text{int}}(t-t_i)} |\Omega\rangle = |0\rangle$$

Therefore we define:

$$U_I(t,-\infty) = Te^{-i\int_{-\infty(1-i\epsilon)}^{t} dt' H_{\text{int}}(t')}$$
$$U_I^\dagger(t,-\infty) = \overline{T} e^{i\int_{-\infty(1+i\epsilon)}^{t} dt' H_{\text{int}}(t')}$$

Here T is the time ordering operator and $\bar{T}$ is the anti-time ordering operator. Hence,

$$\langle \mathcal{O}(t)\rangle = \langle\Omega| \mathcal{O}(t) |\Omega\rangle = \langle 0| U_I^\dagger(t,-\infty)\mathcal{O}_I(t) U_I(t,-\infty) |0\rangle$$
$$= \langle 0| \overline{T} e^{i\int_{-\infty(1+i\epsilon)}^{t} dt' H_{\text{int}}(t')} \mathcal{O}_I(t) T e^{-i\int_{-\infty(1-i\epsilon)}^{t} dt' H_{\text{int}}(t')} |0\rangle$$

The above result is derived in Interaction Picture. If we can use the assumption of adiabatic switching, such that

$$|0\rangle_{\text{out}} = e^{i\theta} |0\rangle_{\text{in}}$$

then, both in-in formalism and in-out formalism coincide.

2.2 Feynman Rules

Rather than using the above mentioned general expression for calculating the correlation function. We will often rely on the diagrammatic approach. The diagrammatic approach to computing in-in correlation functions follows a set of rules that can be stated without explicit derivation. For an interaction Lagrangian of the form

$$\mathcal{L}_{\text{int}} = -\frac{V}{n!}\phi^n,$$

the Feynman rules are summarized as follows:

1. Begin by drawing the final time surface at $t = t_*$.
2. Include all relevant diagrams contributing to the desired process, allowing interaction vertices both above and below the final time surface.
3.
   - Propagators corresponding to lines that intersect or terminate on the final time surface are Wightman functions:
   $$W_k(t,t') = f_k(t) f_k^*(t'),$$
   where the time argument t corresponds to the uppermost vertex and t' to the lowermost.
   - Lines entirely below the final time surface are associated with the Feynman propagator:
   $$G_F(k;t,t') = W_k(t,t')\theta(t-t') + W_k(t',t)\theta(t'-t).$$
   - Lines entirely above the final time surface correspond to time-reversed Feynman propagators:
   $$G_F(k;t',t) = G_F^*(k;t,t').$$
4. Assign a factor of $-iV$ to each vertex below the final time surface and $iV^\dagger$ to each vertex above it. Here, V is determined by functional differentiation of the interaction Hamiltonian. Momentum conservation is imposed at each vertex, along with an overall momentum-conserving delta function in Fourier space.
5. Integrate over the time coordinates of all vertex insertions using the measure $\mathrm{d}t \equiv \mathrm{d}t\, a^3(t)$. For diagrams with internal loops, also integrate over the loop momenta.

6. Include any relevant symmetry factors, dividing the diagram by them where applicable. If diagrams are related by reflection across the final time surface, consider only one representative and take 2 Re of the final result.

These rules will be employed to compute the correlation functions in the context of our analysis.

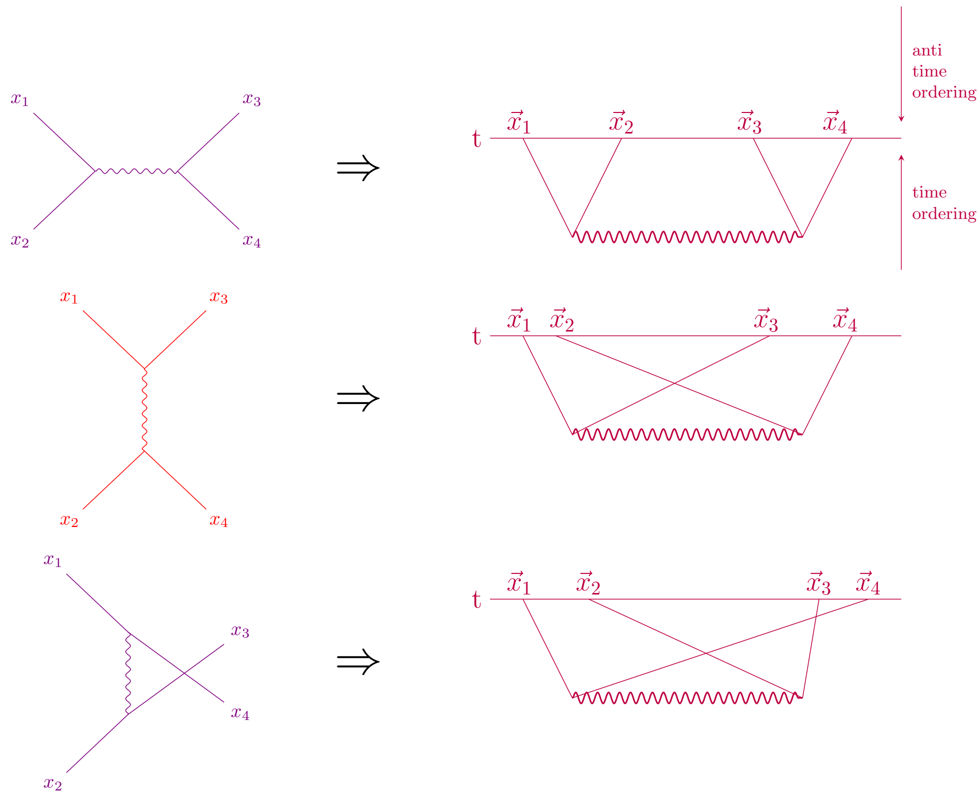


Figure 2.2: We see how the feynman diagrams in the in-out formalism transform into in-out feynman diagram. Note that in the first diagram $x_1 = (t_1, \vec{x}_1)$ and $x_2 = (t_2, \vec{x}_2)$ share a common vertex on both sides, in second diagram, x_1 and x_3 share the same vertex and in the last x_1 and x_4 share the same vertex.

Chapter 3

Quantum Field Theory in curved spacetime

In flat spacetime, quantum field theory provides a well-defined framework for particle physics, with a unique vacuum and global Poincaré symmetry that guarantees energy conservation and a consistent particle interpretation. However, in curved spacetime or non-inertial frames, these assumptions no longer hold. The very notion of particles becomes observer-dependent, and the vacuum state is no longer unique. This breakdown lies at the heart of phenomena such as Unruh and Hawking radiation, and plays a crucial role in the physics of the early universe.

In this chapter, we develop the formal structure necessary to describe quantum fields in curved and accelerating backgrounds. We begin by revisiting canonical quantization of the scalar field in general curved spacetime and introduce the Klein-Gordon inner product to extract a consistent creation and annihilation operator algebra. The ambiguity in the choice of mode functions leads naturally to Bogoliubov transformations, which relate vacua associated with different observers.

To make this abstract discussion concrete, we study Rindler spacetime — the coordinate system appropriate for uniformly accelerating observers — and derive the Unruh effect: the prediction that such observers detect a thermal spectrum even in Minkowski vacuum[9]. The transformation between Minkowski and Rindler modes is worked out explicitly using Bogoliubov coefficients, with a focus on preserving commutation relations. We further explore Killing vector fields as generators of isometries, showing how they can be used to identify conserved quantities and interpret positive-frequency solutions as particle states.

This chapter lays the groundwork for understanding gravitational particle production in an expanding universe, which we explore in detail in the next chapter. By carefully analyzing observer-dependent vacuum structure, we develop the mathematical and physical tools needed to describe particle creation purely due to spacetime dynamics.

3.1 Bogoliubov Transformation

This part is crucial for understanding how particle production modifies the vacuum and which can then impact the observable correlators like the bispectrum. We begin by defining the commutator between the annihilation and creation operators in the context of canonical quantization and limit ourselves to the scalar field in this chapter. Given the solutions f and g to the Klein-Gordon equation, the commutator is defined as:

$$[a(f), a^\dagger(g)] = \langle f, g\rangle,$$

where the norm $\langle f, g\rangle$ represents the inner product of the solutions f and g, and is given by:

$$\langle f, g\rangle = i\int d^3x\sqrt{|h|}\left\{f^*(\partial_t g) - (\partial_t f^*)g\right\}.$$

This expression for the inner product arises from the structure of the Klein-Gordon equation, ensuring that the commutator between the annihilation and creation operators is consistent with the quantization of the field. It is important to note that this inner product is antisymmetric, meaning:

$$\langle g, f\rangle^* = -\langle f, g\rangle.$$

We can also verify this symmetry explicitly as follows:

$$\langle g^*, f^*\rangle = i\int d^3x\sqrt{|h|}\left\{g(\partial_t f^*) - (\partial_t g)f^*\right\}$$

$$= -\langle f, g \rangle.$$

Next, we define the field $\phi(t, x)$ as a sum over the creation and annihilation operators:

$$\phi(t, x) = \sum_{\boldsymbol{k}} \left(a_{\boldsymbol{k}} u_{\boldsymbol{k}} + a^{\dagger}_{\boldsymbol{k}} u^{*}_{\boldsymbol{k}} \right).$$

Using this definition, we can express the annihilation operator $a_{\boldsymbol{k}}$ as the inner product of the field $\phi(t, x)$ with the mode function $u_{\boldsymbol{k}}$:

$$a_{\boldsymbol{k}} = \langle u_{\boldsymbol{k}}, \phi(t, x) \rangle.$$

In curved spacetime or non-inertial coordinates, different observers will disagree on what constitutes a vacuum or particle. This disagreement is encoded mathematically through Bogoliubov transformations, which relate two distinct sets of mode functions. Crucially, if the vacuum defined by one observer appears as a squeezed state to another, this results in a non-zero particle number expectation — a key signature of gravitational particle production.

Much like in classical physics, we studied that canonical transformations preserve the structure of Poisson Brackets, i.e.:

$$\{P, Q\}_{\text{PB}} = \{P', Q'\}_{\text{PB}}$$

A similar concept also exists in quantum field theory for creation and annihilation operators. We start with the condition that:

$$[a, a^{\dagger}] = [b, b^{\dagger}]$$

and the transformation relation which satisfies it

$$\begin{aligned} b &= c_1 a + c_2 a^{\dagger} \\ b^{\dagger} &= d_1 a + d_2 a^{\dagger} \end{aligned}$$

since we know that:

$$\begin{aligned} b^{\dagger} &= (c_1 a + c_2 a^{\dagger})^{\dagger} \\ &= c_1^{\star} a^{\dagger} + c_2^{\star} a \\ \implies d_1 &= c_2^{\star} \\ d_2 &= c_1^{\star} \end{aligned}$$

now

$$\begin{aligned} [b, b^{\dagger}]_{\pm} &= [c_1 a + c_2 a^{\dagger}, c_1^{\star} a^{\dagger} + c_2^{\star} a]_{\pm} \\ &= c_1^{\star}[c_1 a + c_2 a^{\dagger}, a^{\dagger}]_{\pm} + c_2^{\star}[c_1 a + c_2 a^{\dagger}, a]_{\pm} \\ &= c_1^2 [a, a^{\dagger}]_{\pm} + c_1^{\star} c_2 \cancelto{0}{[a^{\dagger}, a^{\dagger}]} + c_2^{\star} c_1 \cancelto{0}{[a, a]} + c_2^2 [a^{\dagger}, a]_{\pm} \\ &= (c_1^2 \mp c_2^2)[a, a^{\dagger}]_{\pm} \end{aligned}$$

where, $\pm$ refers to fermionic or bosonic creation and annihilation operator respectively. Now, we will apply this concept to study Unruh radiation. The goal is this

1. Solve the KG equation in minkowski coordinates
2. Solve the KG equation in rindler coordinates
3. Find the transformation of creation and annihilation operator between the two coordinates which preserves the commutation relationship.

There is another key result that we have to discuss. Given how the modes transform between two coordinates, **how can we find the corresponding transformation between the creation and annihilation operators** in the two coordinate systems?

$$\phi(x) = \sum_i [\hat{a}_i f_i + \hat{a}_i^{\dagger} f_i^{\star}] \qquad\qquad \phi(x) = \sum_i [\hat{b}_i g_i + \hat{b}_i^{\dagger} g_i^{\star}]$$

Let us consider the following transformation between modes:

$$f_i = \sum_j \alpha_{ij} g_j + \beta_{ij} g_j^{\star}$$

then,

$$\begin{aligned}
\phi(x) &= \sum_i [\hat{a}_i f_i + \hat{a}_i^\dagger f_i^\star] \\
&= \sum_i \left[\hat{a}_i \sum_j (\alpha_{ij} g_j + \beta_{ij} g_j^\star) + \hat{a}_i^\dagger \sum_j (\alpha_{ij}^\star g_j^\star + \beta_{ij}^\star g_j) \right] \\
&= \sum_i \left[\sum_j (\alpha_{ij}\hat{a}_i + \beta_{ij}^\star \hat{a}_i^\dagger) g_j + \sum_j (\beta_{ij}\hat{a}_i + \alpha_{ij}^\star \hat{a}_i^\dagger) g_j^\star \right] \\
&= \sum_j \left(\underbrace{\sum_i (\alpha_{ij}\hat{a}_i + \beta_{ij}^\star \hat{a}_i^\dagger)}_{\hat{b}_j} g_j + \underbrace{\sum_i (\beta_{ij}\hat{a}_i + \alpha_{ij}^\star \hat{a}_i^\dagger)}_{\hat{b}_j^\dagger} g_j^\star \right) \\
&= \sum_j [\hat{b}_j g_j + \hat{b}_j^\dagger g_j^\star]
\end{aligned}$$

If we wish to think in terms of matrices then,

$$\begin{bmatrix} f_i \\ f_i^\star \end{bmatrix} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} \\ \beta_{ij}^\star & \alpha_{ij}^\star \end{bmatrix} \begin{bmatrix} g_j \\ g_j^\star \end{bmatrix}$$
$$\begin{bmatrix} \hat{b}_i \\ \hat{b}_i^\dagger \end{bmatrix} = \begin{bmatrix} \alpha_{ji} & \beta_{ji}^\star \\ \beta_{ji} & \alpha_{ji}^\star \end{bmatrix} \begin{bmatrix} \hat{a}_j \\ \hat{a}_j^\dagger \end{bmatrix}$$

Similarly, we consider the inverse transformation:

$$g_i = \sum_j \lambda_{ij} f_j + \mu_{ij} f_j^\star$$

The coefficients can be extracted by utilizing the normalization condition:

$$\begin{aligned}
\langle f_i, f_j \rangle &= \delta_{ij} \\
\langle f_i^\star, f_j^\star \rangle &= -\langle f_i, f_j \rangle^\star = -\delta_{ij}
\end{aligned}$$

The last condition is direct consequence of how Klein Gordon inner product is defined. we get:

$$\begin{aligned}
\langle g_i, f_j \rangle &= \left\langle \sum_k \lambda_{ik} f_k + \mu_{ik} f_k^\star, f_j \right\rangle \\
&= \sum_k \lambda_{ik} \langle f_k, f_j \rangle + \mu_{ik} \cancelto{0}{\langle f_k^\star, f_j \rangle} \\
&= \sum_k \lambda_{ik} \delta_{kj} = \lambda_{ij}
\end{aligned}$$

and

$$\begin{aligned}
\langle g_i, f_j^\star \rangle &= \left\langle \sum_k \lambda_{ik} f_k + \mu_{ik} f_k^\star, f_j^\star \right\rangle \\
&= \sum_k \lambda_{ik} \langle f_k, f_j^\star \rangle + \mu_{ik} \langle f_k^\star, f_j^\star \rangle \\
&= \sum_k -\mu_{ik} \delta_{kj} = -\mu_{ij}
\end{aligned}$$

utilizing

$$\langle a, b \rangle = \langle b, a \rangle^\star = -\langle a^\star, b^\star \rangle$$

we get:

$$\begin{aligned}
\langle g_i, f_j \rangle &= \langle f_j, g_i \rangle^\star = \alpha_{ji}^\star \\
\langle g_i, f_j^\star \rangle &= \langle f_j^\star, g_i \rangle^\star = -\underbrace{\langle f_j, g_i^\star \rangle}_{-\beta_{ji}}
\end{aligned}$$

$$= \beta_{ji}$$

Hence,

$$g_i = \sum_j (\alpha^\star_{ji} f_j - \beta_{ji} f^\star_j)$$

Finally the transformation can be expressed as:

$$\begin{aligned}
\phi(x) &= \sum_i [\hat{b}_i g_i + \hat{b}^\dagger_i g^\star_i] \\
&= \sum_i \left[\hat{b}_i \sum_j (\alpha^\star_{ji} f_j - \beta_{ji} f^\star_j) + \hat{b}^\dagger_i \sum_j (\alpha_{ji} f^\star_j - \beta^\star_{ji} f_j) \right] \\
&= \sum_i \left[\sum_j (\alpha^\star_{ji} \hat{b}_i - \beta^\star_{ji} \hat{b}^\dagger_i) f_j + \sum_j (\alpha_{ji} \hat{b}^\dagger_i - \beta_{ij} \hat{b}_i) f^\star_j \right] \\
&= \sum_j \left(\underbrace{\sum_i (\alpha^\star_{ji} \hat{b}_i - \beta^\star_{ji} \hat{b}^\dagger_i)}_{\hat{a}_j} f_j + \underbrace{\sum_i (\alpha_{ji} \hat{b}^\dagger_i - \beta_{ij} \hat{b}_i)}_{\hat{a}^\dagger_j} f^\star_j \right) \\
&= \sum_j [\hat{a}_j f_j + \hat{a}^\dagger_j f^\star_j]
\end{aligned}$$

In matrix form:

$$\begin{bmatrix} g_i \\ g^\star_i \end{bmatrix} = \begin{bmatrix} \alpha^\star_{ji} & -\beta_{ji} \\ -\beta^\star_{ji} & \alpha_{ji} \end{bmatrix} \begin{bmatrix} f_j \\ f^\star_j \end{bmatrix}$$

$$\begin{bmatrix} \hat{a}_i \\ \hat{a}^\dagger_i \end{bmatrix} = \begin{bmatrix} \alpha^\star_{ij} & -\beta^\star_{ij} \\ -\beta_{ij} & \alpha_{ij} \end{bmatrix} \begin{bmatrix} \hat{b}_j \\ \hat{b}^\dagger_j \end{bmatrix}$$

The above transformation corresponds to bosons, since

$$\alpha^2 - \beta^2 = 1$$

In case of fermions, we have

$$\alpha^2 + \beta^2 = 1$$

but the inner product would also be defined in equivalent manner.

3.2 Rindler Spacetime

The accelerated observer is described by the trajectory:

$$ct(\tau) = \frac{c}{\alpha} \sinh(\alpha\tau) \qquad\qquad x(\tau) = \frac{c}{\alpha} \cosh(\alpha\tau)$$

and, we observe that

$$x^2 - c^2 t^2 = \frac{c^2}{\alpha^2}$$

where α is the norm of 4-acceleration. We can define the new hyperbolic coordinate, more acquainted to study accelerated motion.

$$ct = \frac{c e^{a\xi/c}}{a} \sinh(a\eta) \qquad\qquad x = \frac{c e^{a\xi/c}}{a} \cosh(a\eta) \tag{3.1}$$

Let us for a minute look back at the coordinate transformation. Clearly this covers only the region $x > 0$ and as

$$x - t = \frac{c}{a} e^{a\xi/c} [\cosh(a\eta) - \sinh(a\eta)] = \frac{c}{a} e^{a\xi/c} e^{a\eta} > 0$$

Our coordinate transformation **only covers the region $\mathbf{x} > |\mathbf{t}|$**, not the entire spacetime. To extend the coordinates to all of spacetime, we must also include the other three regions. R II and R III are spacelike, so we won't focus on them here—they can be accessed via analytic continuation if needed. For R IV, we simply flip the signs, yielding:

$$ct = -\frac{ce^{a\xi/c}}{a}\sinh(a\eta), \qquad x = -\frac{ce^{a\xi/c}}{a}\cosh(a\eta), \qquad \text{for } x < |t| \tag{3.2}$$

Strictly speaking, we're abusing notation since both R I and R IV have $\xi, \eta \in (-\infty, +\infty)$, but this will be handled by clearly stating the region under discussion. The origin remains uncovered, but it's a zero-measure point and can be ignored for now. We'll revisit analytic continuation in section 3.4 during our discussion of Bogoliubov transformations.

In the new $(c\eta, \xi)$ coordinates, the metric becomes [1] :

$$\begin{aligned}
ds^2 &= c^2dt^2 - dx^2 \\
&= \left(d\left[\frac{ce^{a\xi/c}}{a}\sinh(a\eta)\right]\right)^2 - \left(d\left[\frac{ce^{a\xi/c}}{a}\cosh(a\eta)\right]\right)^2 \\
&= \left[e^{a\xi/c}\sinh(a\eta)d\xi + ce^{a\xi/c}\cosh(a\eta)d\eta\right]^2 \\
&\quad - \left[e^{a\xi/c}\cosh(a\eta)d\xi + ce^{a\xi/c}\sinh(a\eta)d\eta\right]^2 \\
&= e^{2a\xi/c}\left[c^2d\eta^2 - d\xi^2\right]
\end{aligned}$$

The proper acceleration in these coordinates is:

$$\begin{aligned}
x^2 - c^2t^2 &= \frac{c^2}{\alpha^2} \\
\frac{c^2e^{2a\xi/c}}{a^2}\left[\cosh^2(a\eta) - \sinh^2(a\eta)\right] &= \frac{c^2}{\alpha^2} \\
\Rightarrow \alpha &= ae^{-a\xi/c}
\end{aligned}$$

We can easily spot that the metric tensor is independent of $c\eta$ and thus $\partial_{c\eta} \equiv (1, 0)$ is the killing vector in this coordinate. The norm of this killing vector field is given by

$$\begin{aligned}
\boldsymbol{g}(\partial_\eta, \partial_\eta) &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{bmatrix} g^{\eta\eta} & g^{\eta\xi} \\ g^{\eta\xi} & g^{\xi\xi} \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= g^{\eta\eta} \\
&= e^{2a\xi/c} > 0
\end{aligned} \tag{3.3}$$

This is timelike killing vector. Switching to natural units with $c = 1$. The Klein Gordon Equation of massless scalar field for the Rindler observer is:

$$\begin{aligned}
\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi) &= 0 \\
e^{-2a\xi}\partial_\mu(e^{2a\xi}g^{\mu\nu}\partial_\nu)\phi &= 0 \\
e^{-2a\xi}\partial_\eta(\underbrace{e^{2a\xi}g^{\eta\eta}}_{1}\partial_\eta)\phi + e^{-2a\xi}\partial_\xi(\underbrace{e^{2a\xi}g^{\xi\xi}}_{-1}\partial_\xi)\phi &= 0 \\
e^{-2a\xi}(\partial_\eta^2 - \partial_\xi^2)\phi &= 0
\end{aligned}$$

These are differential equation with constant coefficient, thus the general solution is linear combination of plane wave solutions.

$$\phi = \sum Ae^{-ik_\mu x^\mu} + Be^{ik_\mu x^\mu} = \sum Ae^{-ik_\eta\eta + ik_\xi\xi} + Be^{ik_\eta\eta - ik_\xi\xi}$$

with $k^\mu = (k_\eta, k_\xi)$. The solution to Klein Gordon Equation in minkowski coordinate is given via:

$$\phi = \sum A\, e^{-ik_t t + ik_x x} + B\, e^{ik_t t - ik_x x}$$

[1] With the redefinition $d\bar{\xi} = \left(1 + \frac{a}{c}\bar{\xi}\right)d\xi \Rightarrow \xi = \frac{c}{a}\ln\left(1 + \frac{a}{c}\bar{\xi}\right) \Rightarrow e^{a\xi/c} = 1 + \frac{a}{c}\bar{\xi}$, we get:

$$ds^2 = \left(1 + \frac{a}{c}\bar{\xi}\right)^2 c^2d\eta^2 - d\bar{\xi}^2$$

3.3 Killing Vector Field

In general relativity, killing vector fields are defined as the generator of isometry in the same sense, as we define momentum as generator of translation in Quantum Field Theory. The group theoretic structure of these killing vector fields emerges from their lie algebra. I'd like to remind us that from weinberg's QFT volume 1, we learnt that the eigenvalue of the generator of symmetry is the conserved quantity. Therefore one can expect a version of the same to be true even here. We will use the killing vector field to define the basis, but first we need to revise certain things.

A general coordinate transformation can be expressed as:

$$\begin{aligned} y^\alpha &= e^{\epsilon \mathcal{L}_\xi} x^\alpha = e^{\epsilon\xi} x^\alpha \\ &\approx x^\alpha - \epsilon \xi^\alpha \end{aligned} \qquad (\xi = \xi^\mu \partial_\mu)$$

In the exact above sense, we interpret the killing vector ξ as the generator of isometry i.e. they preserve the norm of 4-vectors. Killing vectors are quite useful in the General Relativity as they lead to certain conserved quantity along the geodesic as follows:

$$\begin{aligned} \frac{\mathrm{d}K_\mu p^\mu}{\mathrm{d}\lambda} &= p^\nu \nabla_\nu (K_\mu p^\mu) \\ &= p^\nu (\nabla_\nu K_\mu) p^\mu + p^\nu K_\mu (\nabla_\nu p^\mu) \\ &= \frac{1}{2} p^\nu p^\mu (\nabla_\nu K_\mu + \nabla_\mu K_\nu) + K_\mu (p^\nu \nabla_\nu p^\mu) \\ &= 0 \end{aligned}$$

The first term vanishes due to Killing equation and the second term due to Geodesic Equation. Another insight can be borrowed from studying the exponential map of killing vector fields in Minkowski spacetime. We begin by first finding out the killing vectors:

$$\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = 0$$

Note that $\Gamma^i_{jk} = 0$ in Minkowski coordinates. For $\mu = \nu = t$

$$\nabla_t \chi_t + \nabla_t \chi_t = 0 \implies \chi_t = f(x)$$

for $\mu = \nu = x$

$$\nabla_x \chi_x + \nabla_x \chi_x = 0 \implies \chi_x = g(t)$$

for $\mu = t$ and $\nu = x$

$$\begin{aligned} \nabla_t \chi_x + \nabla_x \chi_t &= 0 \\ \frac{1}{c}\frac{\partial g(t)}{\partial t} + \frac{\partial f(x)}{\partial x} &= 0 \end{aligned}$$

This implies that the solution are

$$f(x) = -x \qquad\qquad g(t) = ct$$

Therefore, in component form:

$$\chi^\mu = (x, ct, 0, 0) \qquad\qquad \chi_\mu = (-x, ct, 0, 0)$$

or, with $c = 1$

$$\chi = x\partial_t + t\partial_x \tag{3.4}$$

We will use this representation to derive the Lorentz Transformation. Consider the following:

$$\begin{aligned} e^{\beta\chi} x &= \sum_{n=0}^{\infty} \frac{1}{n!} \beta^n \left(x\partial_t + t\partial_x\right)^n x \\ &= x + \beta\left(x\partial_t + t\partial_x\right) x + \frac{\beta^2}{2!}\left(x\partial_t + t\partial_x\right)^2 x + \frac{\beta^3}{3!}\left(x\partial_t + t\partial_x\right)^3 x \ldots \\ &= x + \beta t + \frac{\beta^2}{2!}\left(x\partial_t + t\partial_x\right) t + \frac{\beta^3}{3!}\left(x\partial_t + t\partial_x\right) x \ldots \end{aligned}$$

$$
\begin{aligned}
&= \left(1 + \frac{\beta^2}{2!} + \dots\right) x + \left(\beta + \frac{\beta^3}{3!} + \dots\right) t \\
&= \cosh(\beta) x + \sinh(\beta) t
\end{aligned}
$$

Similarly

$$
\begin{aligned}
e^{\beta\chi} t &= \sum_{n=0}^{\infty} \frac{1}{n!} \beta^n \left(x\partial_t + t\partial_x\right)^n t \\
&= t + \beta \left(x\partial_t + t\partial_x\right) t + \frac{\beta^2}{2!} \left(x\partial_t + t\partial_x\right)^2 t + \frac{\beta^3}{3!} \left(x\partial_t + t\partial_x\right)^3 t \dots \\
&= t + \beta x + \frac{\beta^2}{2!} \left(x\partial_t + t\partial_x\right) x + \frac{\beta^3}{3!} \left(x\partial_t + t\partial_x\right) t \dots \\
&= \left(1 + \frac{\beta^2}{2!} + \dots\right) t + \left(\beta + \frac{\beta^3}{3!} + \dots\right) x \\
&= \cosh(\beta) t + \sinh(\beta) x
\end{aligned}
$$

and

$$
\begin{aligned}
e^{\beta\chi} y &= y \\
e^{\beta\chi} z &= z
\end{aligned}
$$

Represented in matrix form:

$$
\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cosh(\beta) & \sinh(\beta) & 0 & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}
$$

The above transformation describes boost. Since, it is used to jump between the frame of references. The conserved quantity associated with this is:

$$
\chi_\mu p^\mu = -xE + p_x t
$$

is due to $\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = 0$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} \chi_\mu p^\mu &= -E \frac{\mathrm{d}x}{\mathrm{d}\tau} + p_x \frac{\mathrm{d}t}{\mathrm{d}\tau} - x \cancelto{0}{\frac{\mathrm{d}E}{\mathrm{d}\tau}} + t \cancelto{0}{\frac{\mathrm{d}p_x}{\mathrm{d}\tau}} \\
&= -\gamma E v + \gamma^2 m v \\
&= -\gamma^2 m v + \gamma^2 m v && \text{(using } E = \gamma m\text{)} \\
&= 0
\end{aligned}
$$

We have another killing vector field which describes the rotation about z-axis

$$
\xi = (0, -y, x, 0) \equiv -y\partial_x + x\partial_y
$$

The relevant transformation is:

$$
\begin{aligned}
e^{\theta\xi} x &= \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n \left(-y\partial_x + x\partial_y\right)^n x \\
&= x + \theta \left(-y\partial_x + x\partial_y\right) x + \frac{\theta^2}{2!} \left(-y\partial_x + x\partial_y\right)^2 x + \frac{\theta^3}{3!} \left(-y\partial_x + x\partial_y\right)^3 x \dots \\
&= x - \theta y - \frac{\theta^2}{2!} \left(-y\partial_x + x\partial_y\right) y - \frac{\theta^3}{3!} \left(-y\partial_x + x\partial_y\right) x \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \dots\right) x - \left(\theta - \frac{\theta^3}{3!} + \dots\right) y \\
&= \cos(\theta) x - \sin(\theta) y
\end{aligned}
$$

Similarly,

$$
e^{\theta\xi} y = \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n \left(-y\partial_x + x\partial_y\right)^n y
$$

$$
\begin{aligned}
&= y + \theta\left(-y\partial_x + x\partial_y\right)y + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)^2 y + \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)^3 y \dots \\
&= y + \theta x + \frac{\theta^2}{2!}\left(-y\partial_x + x\partial_y\right)x - \frac{\theta^3}{3!}\left(-y\partial_x + x\partial_y\right)y \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \dots\right)y + \left(\theta - \frac{\theta^3}{3!} + \dots\right)x \\
&= \cos(\theta)y + \sin(\theta)x
\end{aligned}
$$

Represented in matrix form:

$$
\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}
$$

The conserved quantity associated with this killing vector is:

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_\mu p^\mu = 0 \implies \xi_\mu p^\mu = -yp_x + xp_y = J_z
$$

Since, the metric in Minkowski spacetime is independent of all coordinates, there are 4 Killing vectors along each of those axis[2]. We define

$$
p^\mu = (E, p_x, p_y, p_z)
$$

$$
\begin{aligned}
K_t &= \partial_t \equiv (1,0,0,0) & K_t p^t &= E \\
K_x &= \partial_x \equiv (0,1,0,0) & K_x p^x &= p_x \\
K_y &= \partial_y \equiv (0,0,1,0) & K_y p^y &= p_y \\
K_z &= \partial_z \equiv (0,0,0,1) & K_z p^z &= p_z
\end{aligned}
$$

The transformation associated with them are:

$$
\begin{aligned}
e^{t'K_t}x(t) &= e^{t'\partial_t}x(t) \\
&= \sum_{n=0}^{\infty} \frac{(t')^n}{n!}\frac{\partial^n x(t)}{\partial t^n} = x(t+t')
\end{aligned}
$$

which is the taylor series expansion of position of particle. The general form of Killing vector field in Minkowski spacetime is given as

$$
\xi_\alpha^{(A)} = c_\alpha^{(A)} + \epsilon_{\alpha\beta}^{(A)} x^\beta
$$

where $x^\mu = \left(x^0, x^1, x^2, x^3\right)$ and the killing vectors associated with translation are:

$$
\begin{aligned}
c_\alpha^{(1)} &= (1,0,0,0) & \epsilon_{\alpha\beta}^{(1)} &= 0 \\
c_\alpha^{(2)} &= (0,1,0,0) & \epsilon_{\alpha\beta}^{(2)} &= 0 \\
c_\alpha^{(3)} &= (0,0,1,0) & \epsilon_{\alpha\beta}^{(3)} &= 0 \\
c_\alpha^{(4)} &= (0,0,0,1) & \epsilon_{\alpha\beta}^{(4)} &= 0
\end{aligned}
$$

These above generators lead to conserved quantities such as Energy and Momentum. The Killing vectors associated with boost and rotation are:

$$
c_\alpha^{(5)} = 0 \qquad \epsilon_{\alpha\beta}^{(5)} = \begin{array}{c} \begin{matrix} x^0 & x^1 & x^2 & x^3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(5)} x^\beta = \left(x^1, -x^0, 0, 0\right)
$$

$$
c_\alpha^{(6)} = 0 \qquad \epsilon_{\alpha\beta}^{(6)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(6)} x^\beta = \left(x^2, -x^1, 0, 0\right)
$$

[2] This is the exact reason why killing vector fields are useful. They don't just provide us the constant of motion but also indicate that there exists a special coordinate system in which the metric is independent of those coordinates

$$c_\alpha^{(7)} = 0 \qquad \epsilon_{\alpha\beta}^{(7)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(7)} x^\beta = \left(x^3, -x^0, 0, 0\right)$$

$$c_\alpha^{(8)} = 0 \qquad \epsilon_{\alpha\beta}^{(8)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(8)} x^\beta = \left(0, x^2, -x^1, 0\right)$$

$$c_\alpha^{(9)} = 0 \qquad \epsilon_{\alpha\beta}^{(9)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(9)} x^\beta = \left(0, x^3, -x^1, 0\right)$$

$$c_\alpha^{(10)} = 0 \qquad \epsilon_{\alpha\beta}^{(10)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad \Rightarrow \epsilon_{\alpha\beta}^{(10)} x^\beta = \left(0, 0, x^3, -x^2\right)$$

The above matrices are the matrix representation of killing vectors which are the generators of Poincaré Group mentioned in quantum field theory textbooks.

3.4 Back to Wave Equation

Reverting back to the KG equation in Rindler coordinate:

$$\begin{aligned} \partial_\eta &= \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x \\ &= e^{a\xi}\left(\cosh(a\eta)\partial_t + \sinh(a\eta)\partial_x\right) \\ &= a(x\partial_t + t\partial_x) \end{aligned}$$

We quickly observe that this is the generator of boost along x-axis. Referring to (3.3) and (3.4), we find that it is timelike killing vector field. Reminding ourselves that in Quantum Field Theory, all these generators are actually operators and thus, it makes sense to consider their eigenfunctions as the preferred basis for calculations and interpretation of solutions. We consider the eigenvalue equation for plane waves in Minkowski spacetime.

$$\begin{aligned} \partial_t A e^{-ik_t t + ik_x x} &= -ik_t A e^{-ik_t t + ik_x x} \\ \partial_t B e^{ik_t t - ik_x x} &= ik_t B e^{ik_t t - ik_x x} \end{aligned}$$

In the unitary representation, the above equation looks like:

$$\begin{aligned} i\partial_t A e^{-ik_t t + ik_x x} &= k_t A e^{ik_t t - ik_x x} \\ i\partial_t B e^{ik_t t - ik_x x} &= -k_t B e^{-ik_t t + ik_x x} \end{aligned}$$

hence, $Be^{ik_t t - ik_x x}$ are the negative energy solution and $Ae^{-ik_t t + ik_x x}$ are positive energy solution. Therefore, it is reasonable to expect that A will be interpreted as 'annihilation' operator which destroys particle and B will be referred as 'creation' operator. For Rindler observer[3]:

$$\begin{aligned} \partial_\eta A e^{-ik_\eta \eta + ik_\xi \xi} &= -ik_\eta A e^{-ik_\eta \eta + ik_\xi \xi} \\ \partial_\eta B e^{ik_\eta \eta - ik_\xi \xi} &= ik_\eta B e^{ik_\eta \eta - ik_\xi \xi} \end{aligned}$$

Thus, based on the argument of identifying the coefficient of positive eigenvalue plane wave solution as the annihilation operator. We interpret A as the annihilation operator and B as the creation operator. Hence,

$$\phi_{\text{minkowski}} = \int_{-\infty}^{\infty} \frac{dk_x}{\sqrt{2\pi(2k_t)}}[a_{k_x} e^{-ik_t t + ik_x x} + a_{k_x}^\dagger e^{ik_t t - ik_x x}]$$

and

$$\phi_{\text{rindler}} = \int_{-\infty}^{\infty} \frac{dk_\xi}{\sqrt{2\pi(2k_\eta)}}[b_{k_\xi} e^{-ik_\eta \eta + ik_\xi \xi} + b_{k_\xi}^\dagger e^{ik_\eta \eta - ik_\xi \xi}]$$

[3]Note that here we are using the eigenbasis of generator of boost for mode expansion

The key idea is this: since the wave equation is conformally invariant and Rindler space is conformally related to Minkowski, we can expand the field in plane waves—just as in flat space. Using lightcone coordinates simplifies things further: positive-frequency modes in Rindler match directly with their flat-space counterparts (and similarly for negative-frequency modes). This lack of mode mixing significantly simplifies the computation of Bogolyubov coefficients. Instead of handling each case separately or computing the full Klein-Gordon inner product, we can isolate modes via a simple Fourier transform. We start by splitting the field into positive- and negative-frequency parts.

$$\phi_{\text{rindler}} = \int_0^\infty \frac{dk_\xi}{\sqrt{2\pi(2k_\eta)}}[b_{k_\xi}e^{-ik_\eta\eta+ik_\xi\xi} + b^\dagger_{k_\xi}e^{ik_\eta\eta-ik_\xi\xi}] + \int_{-\infty}^0 \frac{dk_\xi}{\sqrt{2\pi(2|k_\eta|)}}[b_{k_\xi}e^{ik_\eta\eta+ik_\xi\xi} + b^\dagger_{k_\xi}e^{-ik_\eta\eta-ik_\xi\xi}] \tag{3.5}$$

From the wave equation

$$\begin{aligned}(1/c^2\partial_\eta^2 - \partial_\xi^2)e^{-ik_\eta\eta+ik_\xi\xi} &= 0\\ (-k_\eta^2/c^2 + k_\xi^2)e^{-ik_\eta\eta+ik_\xi\xi} &= 0 \implies k_\eta = |k_\xi c|\end{aligned}$$

Defining $k \equiv |k_\xi|$ and using $u_\pm = \eta \pm \xi$ with $c = 1$, to switch to null/lightcone coordinates:

$$\phi_{\text{rindler}} = \int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_-} + b^\dagger_k e^{iku_-}] + \int_0^\infty \frac{dk}{\sqrt{2\pi(2|k|)}}[b_{-k}e^{-iku_+} + b^\dagger_{-k}e^{iku_+}] \tag{3.6}$$

doing the same $x_\pm = ct \pm x$ with $\omega = |k_x|$ in Minkowski spacetime with signature $(+ - - -)$.

$$\begin{aligned}\phi_{\text{minkowski}} &= \int_{-\infty}^\infty \frac{dk_x}{\sqrt{2\pi(2k_t)}}[a_{k_x}e^{-ik_t t+ik_x x} + a^\dagger_{k_x}e^{ik_t t-ik_x x}]\\ &= \int_{-\infty}^\infty \frac{dk_x}{\sqrt{2\pi(2|k_x|)}}[a_{k_x}e^{-i|k_x c|t+ik_x x} + a^\dagger_{k_x}e^{i|k_x c|t-ik_x x}] \qquad \text{(using } k_t = |k_x c|)\end{aligned}$$

using $\omega = |k_x|$

$$= \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i\omega x_-} + a^\dagger_\omega e^{i\omega x_-}] + \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2|\omega|)}}[a_{-\omega}e^{-i\omega x_+} + a^\dagger_{-\omega}e^{i\omega x_+}]$$

Now comes the chief advantage of this approach mentioned above: since the notion of positive/negative momenta is preserved under the conformal transformation from Minkowski to Rindler space, we can directly identify

$$\begin{aligned}\int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_-} + b^\dagger_k e^{iku_-}] &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i\omega x_-} + a^\dagger_\omega e^{i\omega x_-}]\\ \int_0^\infty \frac{dk}{\sqrt{2\pi(2|k|)}}[b_{-k}e^{-iku_+} + b^\dagger_{-k}e^{iku_+}] &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2|\omega|)}}[a_{-\omega}e^{-i\omega x_+} + a^\dagger_{-\omega}e^{i\omega x_+}]\end{aligned}$$

Peforming the fourier transform:

$$\begin{aligned}&\int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}}e^{i\omega' x_-}\int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_-} + b^\dagger_k e^{iku_-}]\\ &\qquad = \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}}e^{i\omega' x_-}\int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i\omega x_-} + a^\dagger_\omega e^{i\omega x_-}]\\ &\qquad = \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}}\int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i(\omega-\omega')x_-} + a^\dagger_\omega e^{i(\omega+\omega')x_-}]\\ &\qquad = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}}[a_\omega\delta(\omega-\omega') + a^\dagger_\omega\delta(\omega+\omega')]\end{aligned}$$

Since the above mode corresponds to positive frequencies i.e. $\omega' > 0$:

$$\begin{aligned}\frac{a_\omega}{\sqrt{2\omega}} &= \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}}[b_k e^{-iku_-+i\omega' x_-} + b^\dagger_k e^{iku_-+i\omega' x_-}]\\ a_\omega &= \int_{-\infty}^\infty \frac{dx_-}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{\sqrt{2\pi}}\sqrt{\frac{\omega}{k}}[b_k e^{-iku_-+i\omega' x_-} + b^\dagger_k e^{iku_-+i\omega' x_-}]\end{aligned}$$

Since the above mode corresponds to positive frequencies i.e. $\omega' > 0$:

$$\frac{a_\omega}{\sqrt{2\omega}} = \int_{-\infty}^{\infty} \frac{dx_-}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}} [b_k e^{-iku_- + i\omega' x_-} + b_k^\dagger e^{iku_- + i\omega' x_-}]$$
$$a_\omega = \int_{-\infty}^{\infty} \frac{dx_-}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\omega}{k}} [b_k e^{-iku_- + i\omega' x_-} + b_k^\dagger e^{iku_- + i\omega' x_-}]$$

we can express the above entirely in terms of minkowski lightcone coordinate as:

$$\begin{aligned} ds^2_{\text{minkowski}} &= ds^2_{\text{rindler}} \\ dt^2 - dx^2 &= e^{2a\xi}(d\eta^2 - d\xi^2) \\ dx_- dx_+ &= e^{-a(u_- - u_+)} du_- du_+ \end{aligned} \tag{3.7}$$

From, above we see that (since there's no mixing of coordinates)

$$x_- = \frac{e^{-au_-}}{-a}$$
$$x_+ = \frac{e^{au_+}}{a}$$

or, using (3.1)

$$\begin{aligned} x_\pm &= t \pm x \\ &= \frac{e^{a\xi}}{a}[\sinh(a\eta) \pm \cosh(a\eta)] \\ &= \frac{e^{a\xi}}{a}\left[\frac{e^{a\eta} - e^{-a\eta}}{2} \pm \frac{e^{a\eta} + e^{-a\eta}}{2}\right] \\ &= \frac{e^{a\xi}}{a}\left[\frac{e^{a\eta} \pm e^{a\eta}}{2} - \frac{e^{-a\eta} \mp e^{-a\eta}}{2}\right] \\ &= \pm\frac{e^{a(\xi\pm\eta)}}{a} = \pm\frac{e^{\pm a u_\pm}}{a} \end{aligned} \tag{3.8}$$

Thus,

$$a_\omega = \int_{-\infty}^{\infty} \frac{dx_-}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\omega}{k}} \left[b_k e^{-iku_- + i\omega' \frac{e^{-au_-}}{a}} + b_k^\dagger e^{iku_- + i\omega' \frac{e^{-au_-}}{a}}\right]$$

Similarly, the inverse transformation is given as:

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} e^{ik'u_-} \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}} [a_\omega e^{-i\omega x_-} + a_\omega^\dagger e^{i\omega x_-}] \\ &= \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} e^{ik'u_-} \int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}} [b_k e^{-iku_-} + b_k^\dagger e^{iku_-}] \\ &= \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\pi(2k)}} [b_k e^{-i(k-k')u_-} + b_k^\dagger e^{i(k+k')u_-}] \\ &= \int_0^\infty \frac{dk}{\sqrt{2k}} [b_k \delta(k-k') + b_k^\dagger \delta(k+k')] \\ &= \frac{b_{k'}}{\sqrt{2k'}} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{b_k}{\sqrt{2k}} &= \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2\omega)}} [a_\omega e^{-i\omega x_- + iku_-} + a_\omega^\dagger e^{i\omega x_- + iku_-}] \\ b_k &= \int_{-\infty}^{\infty} \frac{du_-}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{k}{\omega}} \left[a_\omega e^{i\omega \frac{e^{-au_-}}{a} + iku_-} + a^\dagger e^{-i\omega \frac{e^{-au_-}}{a} + iku_-}\right] \\ &= \int_0^\infty d\omega \left[a_\omega \sqrt{\frac{k}{\omega}} F(\omega, k) + a_\omega^\dagger \sqrt{\frac{k^\star}{\omega^\star}} F(-\omega, k)\right] \end{aligned} \tag{3.9}$$

using above we can also get:

$$(b_k)^\dagger = b_k^\dagger = \int_0^\infty d\omega \left[a_\omega^\dagger \sqrt{\frac{k^\star}{\omega^\star}} F(-\omega,-k) + a_\omega \sqrt{\frac{k}{\omega}} F(\omega,-k) \right] \tag{3.10}$$

Similarly, performing the Fourier transformation on negative modes, we get:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} e^{ik'u_+} \int_0^\infty & \frac{d\omega}{\sqrt{2\pi(2\omega)}} [a_{-\omega} e^{-i\omega x_+} + a_{-\omega}^\dagger e^{i\omega x_+}] \\
&= \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} e^{ik'u_+} \int_0^\infty \frac{dk}{\sqrt{2\pi(2|k|)}} [b_{-k} e^{-iku_+} + b_{-k}^\dagger e^{iku_+}] \\
&= \int_0^\infty \frac{dk}{\sqrt{2|k|}} [b_{-k}\delta(k-k') + b_{-k}^\dagger \delta(k+k')]
\end{aligned}$$

Since, here $k' < 0$, we get:

$$\begin{aligned}
\frac{b_{-k}^\dagger}{\sqrt{2|k|}} &= \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2|\omega|)}} [a_{-\omega} e^{-i\omega x_+ + iku_+} + a_{-\omega}^\dagger e^{i\omega x_+ + iku_+}] \\
&= \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\sqrt{2\pi(2|\omega|)}} \left[a_{-\omega} e^{-i\omega \frac{e^{au_+}}{a} + iku_+} + a_{-\omega}^\dagger e^{i\omega \frac{e^{au_+}}{a} + iku_+} \right] \\
b_{-k}^\dagger &= \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{k}{\omega}} \left[a_{-\omega} e^{-i\omega \frac{e^{au_+}}{a} + iku_+} + a_{-\omega}^\dagger e^{i\omega \frac{e^{au_+}}{a} + iku_+} \right]
\end{aligned} \tag{3.11}$$

and thus:

$$b_{-k} = \int_{-\infty}^{\infty} \frac{du_+}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{k}{\omega}} \left[a_{-\omega}^\dagger e^{i\omega \frac{e^{au_+}}{a} - iku_+} + a_{-\omega} e^{-i\omega \frac{e^{au_+}}{a} - iku_+} \right]$$

From (3.10), we get

$$b_{-k}^\dagger = \int_0^\infty d\omega \left[a_\omega^\dagger \sqrt{\frac{k^\star}{\omega^\star}} F(-\omega,k) + a_\omega \sqrt{\frac{k}{\omega}} F(\omega,k) \right] \tag{3.12}$$

We observe that (3.11) and (3.12) are same with the identification $a_{-\omega}^\dagger = a_\omega$.

Analytic Continuation

An alternative approach to deriving the Bogoliubov transformation—originally due to Unruh—is to expand the field in a complete set of modes defined over the entire Minkowski spacetime.[10] These modes share the same vacuum as the standard Minkowski modes but describe excitations differently. Crucially, they have a simpler overlap with Rindler modes, making the Bogoliubov coefficients easier to compute.

The key idea is to analytically continue the Rindler modes across all of spacetime and express this continuation in terms of the original Rindler modes. However, care is needed—particularly in Region I, where we have:

$$\partial_\eta = a(x\partial_t + t\partial_x)$$

whereas in Region IV, we have

$$\partial_\eta = -a(x\partial_t + t\partial_x)$$

Therefore,

$$\phi(x) = \begin{cases} \int \frac{dk_\xi}{\sqrt{2\pi(2k_\eta)}} [b^{(1)} \underbrace{e^{-ik_\eta \eta + ik_\xi \xi}}_{\sqrt{4\pi k_\eta} g^{(1)}} + b^{(1)\dagger} \underbrace{e^{+ik_\eta \eta - ik_\xi \xi}}_{\sqrt{4\pi k_\eta} g^{(1)\star}}] & \text{In region } I \\ \int \frac{dk_\xi}{\sqrt{2\pi(2k_\eta)}} [b^{(2)} \underbrace{e^{ik_\eta \eta + ik_\xi \xi}}_{\sqrt{4\pi k_\eta} g^{(2)}} + b^{(2)\dagger} \underbrace{e^{-ik_\eta \eta - ik_\xi \xi}}_{\sqrt{4\pi k_\eta} g^{(2)\star}}] & \text{In region } IV \end{cases} \tag{3.13}$$

As we will see, this approach is similar to the one described in section 3.1. We start by writing the solution in Minkowski spacetime

$$\phi(x)_{\text{minkowski}} = \int dk [a_k f + a_k^\dagger f^\star]$$

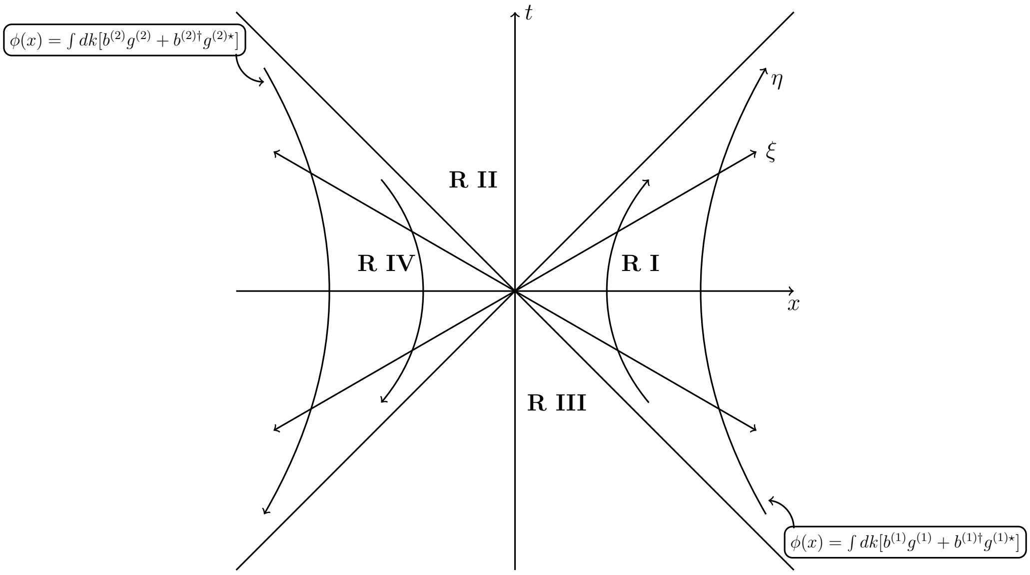


Figure 3.1: We can write the field in the region of overlap ($x > |t|$) of Minkowski space and Rindler space using complete basis set as $\phi(x) = \int dk[b^{(1)}g^{(1)} + b^{(1)\dagger}g^{(1)\star} + b^{(2)}g^{(2)} + b^{(2)\dagger}g^{(2)\star}]$.

In Rindler spacetime, the solution in the combined region I and region IV is written as:

$$\phi(x)_{\text{rindler}} = \int dk[\, \underbrace{b^{(1)}g^{(1)} + b^{(1)\dagger}g^{(1)\star}}_{\text{defined in region 1}} \; + \; \underbrace{b^{(2)}g^{(2)} + b^{(2)\dagger}g^{(2)\star}}_{\text{defined in region 4}} \,]$$

In region I:

$$\begin{aligned} a(x-t) &= e^{a\xi}(\cosh a\eta - \sinh a\eta) \\ &= e^{a\xi}\left(\frac{e^{a\eta}+e^{-a\eta}}{2} - \frac{e^{a\eta}-e^{-a\eta}}{2}\right) \\ &= e^{-a(\eta-\xi)} \\ a(x+t) &= e^{a\xi}(\cosh a\eta + \sinh a\eta) \\ &= e^{a\xi}\left(\frac{e^{a\eta}+e^{-a\eta}}{2} + \frac{e^{a\eta}-e^{-a\eta}}{2}\right) \\ &= e^{a(\eta+\xi)} \end{aligned}$$

In region IV, using (3.2):

$$\begin{aligned} a(-x+t) &= -e^{a\xi}(-\cosh a\eta + \sinh a\eta) \\ &= e^{a\xi}\left(\frac{e^{a\eta}+e^{-a\eta}}{2} - \frac{e^{a\eta}-e^{-a\eta}}{2}\right) \\ &= e^{-a(\eta-\xi)} \\ a(-x-t) &= -e^{a\xi}(-\cosh a\eta - \sinh a\eta) \\ &= e^{a\xi}\left(\frac{e^{a\eta}+e^{-a\eta}}{2} + \frac{e^{a\eta}-e^{-a\eta}}{2}\right) \\ &= e^{a(\eta+\xi)} \end{aligned}$$

A quick observation tells us that region I and region IV are related via a transformation of the form $\eta \to \eta \pm i\pi$. Since, the solution with $k_\eta > 0$ is only convergent in the region where $\text{Im}\{\eta - \xi\} < 0$, we have to take the branch cut in the upper half plane. Therefore, the function $g^{(1)}_{k_\xi}$ is analytic only in the lower half plane which implies the transformation to be considered has to be $\eta \to \eta - i\pi$. Alternatively, If we now assume $k_\eta > 0$ and so $k_\eta = k_\xi c \equiv k$ then we can write for $g^{(1)}_{k_\xi}$ in region I:

$$\begin{aligned} \sqrt{4\pi k_\eta}\, g^{(1)}_{k_\xi} &= e^{-ik_\eta\eta + ik_\xi\xi} = e^{-ik_\xi(\eta-\xi)} = \left[e^{-a(\eta-\xi)}\right]^{ik/a} = [a(x-t)]^{ik/a} \\ &= a^{ik/a}(x-t)^{ik/a} \end{aligned} \tag{3.14}$$

In region IV

$$\sqrt{4\pi k_\eta} g^{(2)\star}_{-k_\xi} = e^{-ik_\eta \eta + ik_\xi \xi} = e^{-ik_\xi(\eta-\xi)} = \left[e^{-a(\eta-\xi)}\right]^{ik/a} = [a(-x+t)]^{ik/a}$$
$$= [ae^{\ln(-x+t)}]^{ik/a}$$

Since the above expression involves $z^{ik/n}$ and the exponent can be non-integer. It is a multivalued function and therefore we have to choose the branch-cut before we proceed.

$$\sqrt{4\pi k_\eta} g^{(2)}_{-k_\xi} = [ae^{\lim_{\epsilon\to 0} \ln\{x-(t-i\epsilon)\}-i\pi}]^{ik/a} = [ae^{\ln|x-t|-i\pi}]^{ik/a} = a^{ik/a}(x-t)^{ik/a} e^{\frac{\pi k}{a}}$$

In the above, we made the choice of branch cut in the upper half of the complex plane therefore, we had to consider $t \to \lim_{\epsilon \to 0}(t - i\epsilon)$ i.e. approach the real t axis from below for the sake of convergence.

The choice of branch cut controls weather we get $e^{\frac{\pi k}{a}}$ or $e^{-\frac{\pi k}{a}}$ in the above equation. Since we considered the branch cut in the upper half of the complex plane, we are effectively doing the analytic continuation in the lower half. The linear combination:

$$\sqrt{4\pi k_\eta}[g^{(1)}_{k_\xi} + e^{-\frac{\pi k}{a}} g^{(2)\star}_{-k_\xi}] = 2a^{ik/a}(x-t)^{ik/a}$$

is similar to (3.14) and is to be interpreted as the analytic continuation of $g^{(1)}_{k_\xi}$. We can consider a normalized version with $k_\eta = k_\xi \equiv k$:

$$h^{(1)}_k = \frac{1}{\sqrt{2\sinh\left(\frac{\pi k}{a}\right)}}\left[e^{\frac{\pi k}{2a}} g^{(1)}_k + e^{\frac{\pi k}{a}} e^{-\frac{\pi k}{2a}} g^{(1)}_k\right]$$
$$= \frac{1}{\sqrt{2\sinh\left(\frac{\pi k}{a}\right)}}\left[e^{\frac{\pi k}{2a}} g^{(1)}_k + e^{-\frac{\pi k}{2a}} g^{(2)\star}_{-k}\right]$$

Coming back to Minkowski coordinate

$$\phi_{\text{minkowski}} = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi(2\omega)}}[a_\omega e^{-i\omega(t-x)} + a^\dagger_\omega e^{i\omega(t-x)}]$$

We observe that $e^{-i\omega(t-x)}$ solution with $\omega > 0$ is only convergent in the region where $\text{Im}\{t-x\} < 0$. Since it is convergent in this region, we need to ensure that it is also analytic here, which means we put the branch cut in the upper half complex plane.[4]

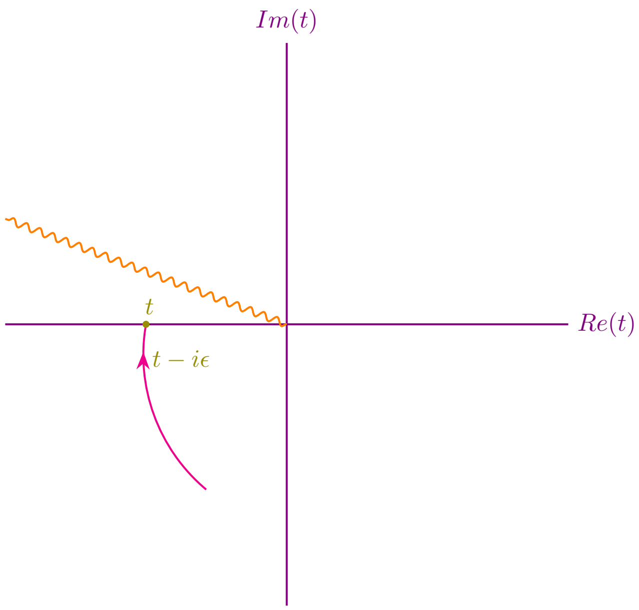


Figure 3.2: Analytic continuation is viewed here as applying transformations that extend or shift the region of convergence. The extended solutions are patched together to form a global description (an atlas). Choosing the branch cut in the upper half of the complex t-plane, we approach the real axis via the analytic region as: $\lim_{\epsilon\to 0}(t - i\epsilon)$

Same procedure can also be applied to find the conjugate mode. In region I for $k_\eta > 0$:

$$\sqrt{4\pi k_\eta} g^{(1)\star}_{-k_\xi} = e^{ik_\eta \eta + ik_\xi \xi} = e^{ik_\xi(\eta+\xi)} = \left[e^{a(\eta+\xi)}\right]^{ik/a} = [a(x+t)]^{ik/a}$$
$$= a^{ik/a}(x+t)^{ik/a} \tag{3.15}$$

In region IV

$$\sqrt{4\pi k_\eta} g^{(2)}_{k_\xi} = e^{ik_\xi \eta + ik_\xi \xi} = e^{ik_\eta(\eta+\xi)} = \left[e^{a(\eta+\xi)}\right]^{ik/a} = [a(-x-t)]^{ik/a}$$
$$= [ae^{\lim_{\epsilon\to 0} \ln\{-x-(t-i\epsilon)\}}]^{ik/a} = [ae^{\ln|x+t|+i\pi}]^{ik/a} = a^{ik/a} e^{-\pi k/a}(x+t)^{ik/a}$$

Thus,

$$h^{(2)}_k = \frac{1}{\sqrt{2\sinh\left(\frac{\pi k}{a}\right)}}\left[e^{\frac{\pi k}{2a}} g^{(2)}_k + e^{\frac{\pi k}{a}} e^{-\frac{\pi k}{2a}} g^{(2)}_k\right]$$
$$= \frac{1}{\sqrt{2\sinh\left(\frac{\pi k}{a}\right)}}\left[e^{\frac{\pi k}{2a}} g^{(2)}_k + e^{-\frac{\pi k}{2a}} g^{(1)\star}_{-k}\right]$$

One could wonder, what about region II and region III? Actually, region II and region III are spacelike, so we aren't worried about them. Also, we expect the fields to commute in that region for the sake of causality.

[4] $\ln z = \ln re^{i\theta} = \ln r + i\theta$ has a branch cut; as $z(r;\theta+2\pi) = z(r;\theta)$ but $\ln z(r;\theta+2\pi) = \ln z(r;\theta) + 2\pi$, here periodicity of z and $\ln z$ aren't same. It is traditional to put this branch cut on the real axis, $\theta = 0$, but we can of course put it anywhere starting at the origin. The important point being that if we rotate θ by 2π we cross the branch cut.

3.5 Relativity of Vacuum

In this section, we will evaluate the number of particles between Minkowski observer and Rindler observer. We start by noting that $b_k^{(1)}\left|0\right\rangle_M \neq 0$ since, $\left|0\right\rangle_M$ is the minkowski vacuum. The way to think about it is this, $b_k^{(1)}$ is the ladder operator corresponding to generator of boost, whereas $a_k^{(1)}$ is the ladder operator corresponding to generator of time translation. Therefore, the shift in eigenvectors that ladder operators are supposed to do is only for their respective eigenstates. We use the notation ${}_M\left\langle 0\right|\ldots\left|0\right\rangle_M \equiv \left\langle\ldots\right\rangle_M$, where M denotes minkowski.

$$\begin{aligned}
\langle N_k\rangle &= \left\langle b_k^\dagger b_k\right\rangle_M \\
&= \int d\omega' d\omega \quad {}_M\left\langle 0\right|\left[a_{\omega'}^\dagger\sqrt{\frac{k^\star}{\omega^{\star\prime}}}F(-\omega',-k) + a_{\omega'}\sqrt{\frac{k}{\omega'}}F(\omega',-k)\right] \\
&\qquad\qquad\qquad \times\left[a_\omega\sqrt{\frac{k}{\omega}}F(\omega,k) + a_\omega^\dagger\sqrt{\frac{k^\star}{\omega^\star}}F(-\omega,k)\right]\left|0\right\rangle_M
\end{aligned}$$

Only non zero contribution comes from:

$$\begin{aligned}
&= \int d\omega' d\omega\sqrt{\frac{k}{\omega'}}\sqrt{\frac{k^\star}{\omega^\star}}F(\omega',-k)F(-\omega,k)\underbrace{\left\langle a_{\omega'}a_\omega^\dagger\right\rangle}_{\delta(\omega'-\omega)} \\
&= \int d\omega\left|\frac{k}{\omega}\right||F(-\omega,k)|^2
\end{aligned}$$

Now, we focus on integrating $F(-\omega,k)$:

$$\begin{aligned}
F(\omega,k) &= \int_{-\infty}^{\infty}\frac{du_-}{2\pi}\exp\left(i\omega\frac{e^{-au_-}}{a} + iku_-\right) \\
&= \int_{-\infty}^{\infty}\frac{du_-}{2\pi}\exp\left(i\omega\frac{e^{-au_-}}{a}\right)e^{iku_-}
\end{aligned}$$

making the substitution $x = e^{-au_-}$, we get

$$\begin{aligned}
-ae^{-au_-}du_- &\to dx \\
\int_{-\infty}^{\infty} &\to \int_{\infty}^{0} \\
\exp\left(i\omega\frac{e^{-au_-}}{a}\right) &\to e^{i\omega x/a} \\
e^{iku_-} &\to [e^{-au_-}]^{\frac{-ik}{a}}
\end{aligned}$$

Therefore

$$\begin{aligned}
F(\omega,k) &= \frac{1}{2\pi}\int_0^\infty\frac{dx}{ax}e^{i\omega x/a}x^{-ik/a} \\
&= \int_0^\infty\frac{dx}{2\pi a}x^{-ik/a-1}e^{i\omega x/a}
\end{aligned}$$

expressing in more familiar form:

$$F(\omega,k) = \int_0^\infty\frac{dx}{2\pi a}x^{s-1}e^{bx}, \qquad s = -\frac{ik}{a}, \qquad b = -\frac{i\omega}{a}.$$

We can utilize the identity

$$\int_0^\infty dx x^{s-1}e^{-bx} = e^{-s\ln(b)}\Gamma(s) \qquad \text{(with } \mathrm{Re}\{b\} > 0 \text{ and } \mathrm{Re}\{s\}\in(0,1))$$

Since, our s and b are purely imaginary, we will use shift them along real axis ($\epsilon > 0$) and then take the limit $\epsilon \to 0$.

$$F(\omega,k) = \lim_{\epsilon\to 0}\int_0^\infty\frac{dx}{2\pi a}x^{(s+\epsilon)-1}e^{(b+\epsilon)x}$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi a} e^{-(s+\epsilon)\ln(b+\epsilon)} \Gamma(s+\epsilon)$$

Since, logarithm is multivalued function in complex plane. We consider the branch cut along $-ve$ x-axis:

$$\begin{aligned}
\lim_{\epsilon \to 0} \ln(b+\epsilon) &= \lim_{\epsilon \to 0} \ln \left| -i\frac{\omega}{a} + \epsilon \right| + \lim_{\epsilon \to 0} i \arg\left(-i\frac{\omega}{a} + \epsilon\right) \\
&= \ln \left|\frac{\omega}{a}\right| + \lim_{\epsilon \to 0} i \tan^{-1}\left(\frac{-\omega/a}{\epsilon}\right) \\
&= \ln \left|\frac{\omega}{a}\right| - i\frac{\pi}{2}\,\mathrm{sgn}\left(\frac{\omega}{a}\right)
\end{aligned}$$

Therefore, we get:

$$\begin{aligned}
F(\omega, k) &= \frac{1}{2\pi a} e^{-s \lim_{\epsilon \to 0} \ln(b+\epsilon)} \Gamma(s) \\
&= \frac{1}{2\pi a} \exp\left[i\frac{k}{a} \left\{ \ln\left|\frac{\omega}{a}\right| - i\frac{\pi}{2}\mathrm{sgn}\left(\frac{\omega}{a}\right) \right\} \right] \Gamma\left(-\frac{i\omega}{a}\right) \\
&= \frac{1}{2\pi a} \exp\left[i\frac{k}{a} \ln\left|\frac{\omega}{a}\right| + \frac{\pi k}{2a}\mathrm{sgn}\left(\frac{\omega}{a}\right) \right] \Gamma\left(-\frac{i\omega}{a}\right)
\end{aligned}$$

Since, we had already defined $a > 0$, assuming $\omega > 0$:

$$\begin{aligned}
F(\omega, k) &= \frac{1}{2\pi a} \exp\left[i\frac{k}{a} \ln\left|\frac{\omega}{a}\right| + \frac{\pi k}{2a} \right] \Gamma\left(-\frac{i\omega}{a}\right) \\
&= \frac{1}{2\pi a} \exp\left[i\frac{k}{a} \ln\left|\frac{\omega}{a}\right| + \frac{\pi k}{a} - \frac{\pi k}{2a} \right] \Gamma\left(-\frac{i\omega}{a}\right)
\end{aligned}$$

since,

$$\begin{aligned}
F(-\omega, k) &= \frac{1}{2\pi a} \exp\left[i\frac{k}{a} \ln\left|\frac{\omega}{a}\right| - \frac{\pi k}{2a} \right] \Gamma\left(-\frac{i\omega}{a}\right) \\
F(\omega, k) &= F(-\omega, k) e^{\pi k/a} \qquad (3.16)
\end{aligned}$$

Here we have two choice, either evaluate the integral explicitly or use an alternative trick by utilizing (3.9):[5]

$$\begin{aligned}
[b_k, b_{k'}^\dagger] = &\int_0^\infty d\omega \int_0^\infty d\omega' \sqrt{\frac{kk'^\star}{\omega\omega'^\star}} \{F(\omega, k)F(-\omega', -k') \\
&- F(-\omega, k)F(\omega', -k')\} \underbrace{[a_\omega, a_{\omega'}^\dagger]}_{\delta(\omega-\omega')} \\
= &\ \delta(k - k')
\end{aligned}$$

using (3.16) and setting $k = k'$

$$\begin{aligned}
\delta(0) &= \int_0^\infty d\omega \left|\frac{k}{\omega}\right| \left[e^{2\pi k/a} |F(-\omega, k)|^2 - |F(-\omega, k)|^2 \right] \\
\int_0^\infty d\omega \left|\frac{k}{\omega}\right| |F(-\omega, k)|^2 &= \frac{\delta(0)}{e^{2\pi k/a} - 1}
\end{aligned}$$

Thus,

$$\langle N_k \rangle = \frac{\delta(0)}{e^{2\pi k/a} - 1}$$

We can absorb the delta function by using hard cutoff as:

$$\begin{aligned}
\delta(k - k') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dx e^{i(k-k')x} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{L \to \infty} \int_{-L}^L dx e^{i(k-k')x} \frac{1}{\sqrt{2\pi}}
\end{aligned}$$

[5] The canonical quantization condition over creation and annihilation operators depends on the convention used for Fourier transform.

at $k = k'$

$$\delta(0) = \frac{V}{2\pi}$$

and then express the particle number in terms of number density.

$$\langle n_k \rangle = \frac{2\pi}{e^{2\pi k/a} - 1} = \frac{2\pi}{e^{E/T} - 1}$$

where $E = \sqrt{k^2 + m^2}|_{m=0}$ and we identify $T = {}^{a}/_{2\pi}$. Alternatively, we could also derive the same calculation using analytic continuation. Let us expand the field in terms the analytically continued basis set:

$$\phi(x) = \int dk[c_k^{(1)} h_k^{(1)} + c_k^{(1)\dagger} h_k^{(1)\star} + c_k^{(2)} h_k^{(2)} + c_k^{(2)\dagger} h_k^{(2)\star}]$$

The same was originally expressed in terms of rindler coordinates as:

$$\phi(x) = \int dk[b_k^{(1)} g_k^{(1)} + b_k^{(1)\dagger} g_k^{(1)\star} + b_k^{(2)} g_k^{(2)} + b_k^{(2)\dagger} b_k^{(2)\star}]$$

Therefore, we consider the bogoliubov transformation which connects the $b_k^{(1)}$ and $b_k^{(2)}$ with $c_k^{(1)}$ and $c_k^{(2)}$. Since the transformation matrix considered here is real and symmetric, we can rewrite the relationship between the creation and annihilation as if:

$$\begin{bmatrix} g_k^{(1)} \\ g_{-k}^{(2)\star} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} h_k^{(1)} \\ h_{-k}^{(2)\star} \end{bmatrix}$$

then,

$$\begin{bmatrix} b_k^{(1)} \\ b_{-k}^{(2)\dagger} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} c_k^{(1)} \\ c_{-k}^{(2)\dagger} \end{bmatrix}$$

Earlier we derived:

$$\begin{bmatrix} h_k^{(1)} \\ h_{-k}^{(2)\star} \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2\sinh({}^{\pi k}/_{a})}} \begin{bmatrix} e^{\pi k/2a} & e^{-\pi k/2a} \\ e^{-\pi k/2a} & e^{\pi k/2a} \end{bmatrix}}_{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1}} \begin{bmatrix} g_k^{(1)} \\ g_{-k}^{(2)\star} \end{bmatrix}$$

Therefore, we get:

$$\begin{bmatrix} b_k^{(1)} \\ b_{-k}^{(2)\dagger} \end{bmatrix} = \frac{1}{\sqrt{2\sinh({}^{\pi k}/_{a})}} \begin{bmatrix} e^{\pi k/2a} & e^{-\pi k/2a} \\ e^{-\pi k/2a} & e^{\pi k/2a} \end{bmatrix} \begin{bmatrix} c_k^{(1)} \\ c_{-k}^{(2)\dagger} \end{bmatrix}$$

Now, we can evaluate the number density of $b_k^{(1)}$ modes noting that $b_k^{(1)} |0\rangle_R = 0$ where $|0\rangle_R$ is the rindler vacuum. But $b_k^{(1)} |0\rangle_M \neq 0$ unless $|0\rangle_M$ and $|0\rangle_R$ coincide. We note that $c_k^{(1)} |0\rangle_M = 0$, then:

$$\begin{aligned}
\left\langle N_k^{(1)} \right\rangle &= {}_M \langle 0| b_k^{(1)\dagger} b_k^{(1)} |0\rangle_M \\
&= \frac{1}{2\sinh(\frac{\pi k}{a})} {}_M \langle 0| (e^{\pi k/2a} c_k^{(1)\dagger} + e^{-\pi k/2a} c_{-k}^{(2)})(e^{\pi k/2a} c_k^{(1)} + e^{-\pi k/2a} c_{-k}^{(2)\dagger}) |0\rangle_M \\
&= \frac{1}{2\sinh(\frac{\pi k}{a})} {}_M \langle 0| e^{-\pi k/2a} c_{-k}^{(2)\dagger} e^{-\pi k/2a} c_{-k}^{(2)} |0\rangle_M \\
&= \frac{e^{-\pi k/a}}{2\sinh({}^{\pi k}/_{a})} {}_M \langle 0| \underbrace{c_{-k}^{(2)\dagger} c_{-k}^{(2)}}_{c_{-k}^{(2)} c_{-k}^{(2)\dagger} + [c_{-k}^{(2)}, c_{-k}^{(2)\dagger}]} |0\rangle_M \\
&= \frac{e^{-\pi k/a}}{2\sinh({}^{\pi k}/_{a})} {}_M \langle 0| \delta(0) |0\rangle_M \\
&= \frac{e^{-\pi k/a}}{e^{\pi k/a} - e^{-\pi k/a}} \delta(0) \\
&= \frac{\delta(0)}{e^{2\pi k/a} - 1}
\end{aligned}$$

where we have used $[c_k^{(1)}, c_k^{(2)}] = 0$, because it can be shown that $\left\langle h_k^{(1)}, h_k^{(2)} \right\rangle = 0$. The above result is to be thought of as, average value of occupation number of rindler modes $b_k^{(1)}$ in minkowski vacuum with the assumption that each observer is sensitive to only one kind of modes.

3.6 Hawking Radiation

We start from Eddington Finkelstein coordinate with a slight difference in how we define the tortoise coordinate.

$$r^\star = r + 2GM \ln\Big(\frac{r}{2GM} - 1\Big) - 2GM$$

The metric with this redefinition, $u = t - r^\star$ and $v = t + r^\star$, takes the form of:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right) dudv + r^2 d\Omega \\ &= -\frac{2GM}{r}\left(\frac{r}{2GM} - 1\right) dudv + r^2 d\Omega \\ &= -\frac{2GM}{r} e^{\ln(r/2GM - 1)} dudv + r^2 d\Omega \\ &= -\frac{2GM e^{1-r/2GM}}{r} e^{v-u/4GM} dudv + r^2 d\Omega \end{aligned} \tag{3.17}$$

If we define[6]:[11]

$$dU = e^{-u/4GM} du \qquad\qquad dV = e^{v/4GM} dv$$

we get:

$$ds^2 = -\frac{2GM}{r} e^{1-r/2GM} dUdV + r^2 d\Omega^2 \tag{3.18}$$

In the near horizon limit:

$$ds^2 = dUdV$$

so the Kruskal vacuum is the appropriate one for an observer sitting next to the black hole horizon. On the other hand, since Kruskal–Szekeres coordinates cover the whole of spacetime, they correspond to the Minkowski vacuum that we studied in the quantization of a scalar field in Rindler space. In the asymptotic limit:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega$$

Therefore, we assume the existence of two observer, one locally inertial observer and the other static or accelerated observer near the horizon. From (3.17) and (3.18), we have:

$$-\frac{2GM}{r} e^{1-r/2GM} e^{v-u/4GM} dudv + r^2 d\Omega = -\frac{2GM}{r} e^{1-r/2GM} dUdV + r^2 d\Omega^2$$

Near the horizon,

$$e^{v-u/4GM} dudv + r^2 d\Omega = dUdV + r^2 d\Omega^2$$

The above expression is similar to (3.7), therefore, the rest of the steps are same as in Unruh radiation with the replacement

$$a = \frac{1}{4GM} \qquad u_+ = v \qquad u_- = u$$
$$x_+ = V \qquad x_- = U$$

in equation (3.8). Therefore, the temperature of the black hole is then, given as:

$$T = \frac{1}{8\pi GM}$$

In this chapter, we only studied the field theory in curved spacetime where we had, atleast locally, a timelike killing vector and still we observed that vacuum was not invariant. In the next chapter we will learn the same physics but in the absence of timelike killing vector.

[6] In the popular literature, it is often defined like $dU = \frac{-1}{4GM} e^{-u/4GM} du$ and $dV = \frac{1}{4GM} e^{v/4GM} dv$

Chapter 4

Gravitational Particle Production

In quantum field theory on Minkowski spacetime, Poincaré symmetry is crucial for defining the vacuum state, which forms the basis for constructing physical states. The vacuum in flat spacetime is defined as a translationally invariant state that is annihilated by all generators of the Poincaré group.[12] However, in curved spacetime, especially that of expanding universe, this definition does not hold because there is no global timelike Killing vector field. This means we need to find another way to define the vacuum.

One widely accepted method is based on the instantaneous diagonalization of the Hamiltonian.[13] The free field is expanded as

$$\phi_k(\eta) = f_k(\eta)a_k + f_k^*(\eta)a_k^\dagger,$$

where $f_k(\eta)$ are the mode functions. To define the vacuum, this mode decomposition is substituted into the Hamiltonian. At each instant, the Hamiltonian is diagonalized, and the mode functions that solve the equations of motion while minimizing the eigenvalue of the Hamiltonian are chosen. The operator coefficients corresponding to these mode functions are interpreted as annihilation operators and its hermitian conjugate the creation operator.

The state annihilated by annihilation operator, associated with the selected mode, defines the vacuum state at that instant. The creation operator corresponding to the same mode is then used to construct a basis set spanned by the current vacuum. All other states are generated by the action of the creation operator on the vacuum state.

This framework provides a natural way to discuss gravitational particle production, where the absence of global symmetry in curved spacetime leads to effects like particle creation from the vacuum. A useful analogy is the forced harmonic oscillator, where the dynamic spacetime background acts as an external driving force. For instance, the Bernard and Duncan model illustrates how gravity can create particles, offering insights into the interaction between quantum fields and curved spacetime.

4.1 Forced Harmonic Oscillator

We will briefly review the key features of Forced Harmonic Oscillator which makes them relevent for understanding particle production in cosmological scenario. Similar technique is employed for studying the gravitational particle production. The Hamiltonian for forced harmonic oscillator in quantum mechanics is given as

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 - J(t)x$$

The equation of motion becomes:

$$\ddot{x} + \omega^2 x = J(t)$$

In quantum mechanics, the coordinate x and the momentum p_x take the role of operators $\hat{x}$ and $\hat{p}_x$ satisfying the equal time commutation relation $[\hat{x}, \hat{p}_x] = i$. The equation of motion then, in Heisenberg Picture is given as:

$$i\hbar\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = [\hat{x}, H] = \hat{p}$$
$$i\hbar\frac{\mathrm{d}\hat{p}_x}{\mathrm{d}t} = [\hat{p}_x, H] = -\omega^2\hat{x} + J(t)$$

Similar to free harmonic oscillator treatment in quantum mechanics, we introduce the following ladder operator (also known as creation and annihilation operator in field theory) as:

$$a = \sqrt{\frac{\omega}{2}}\left[\hat{x} + \frac{i}{\omega}\hat{p}\right]$$
$$a^\dagger = \sqrt{\frac{\omega}{2}}\left[\hat{x} - \frac{i}{\omega}\hat{p}\right]$$

They satisy

$$[a, a^\dagger] = i$$

at each instant of time t. Now, the equation of motion for a and $a^\dagger$ is given as:

$$\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = -i\omega\hat{a} + \frac{i}{\sqrt{2\omega}}J(t)$$
$$\frac{\mathrm{d}\hat{a}^\dagger}{\mathrm{d}t} = i\omega\hat{a}^\dagger - \frac{i}{\sqrt{2\omega}}J(t)$$

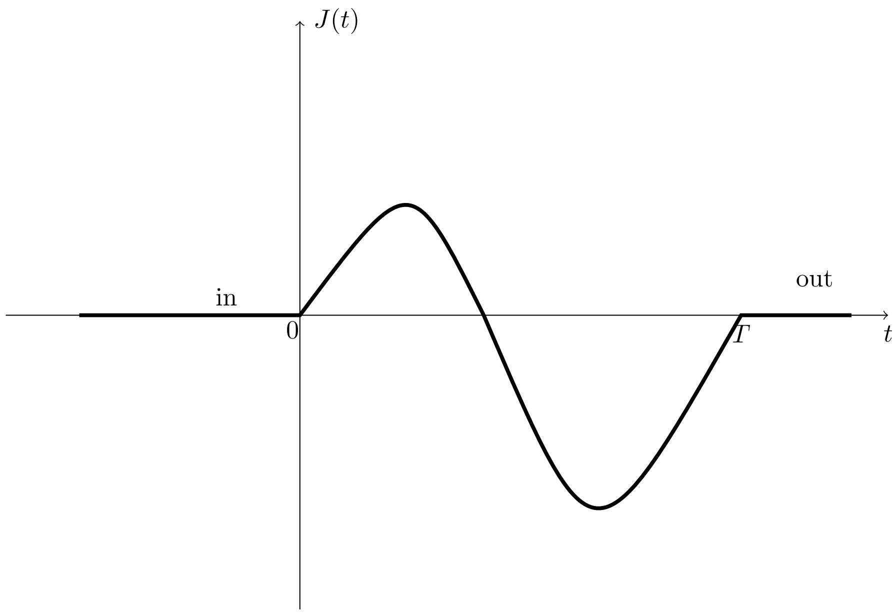
They can be integrated to give:

$$\hat{a}(t) = \left[\hat{a}(t=0) + \frac{i}{\sqrt{2\omega}}\int_0^t e^{i\omega t'}J(t')dt'\right]e^{-i\omega t}$$
$$\hat{a}^\dagger(t) = \left[\hat{a}^\dagger(t=0) - \frac{i}{\sqrt{2\omega}}\int_0^t e^{-i\omega t'}J(t')dt'\right]e^{i\omega t}$$

where $\hat{a}(t=0)$ and $\hat{a}^\dagger(t=0)$ are operator valued constants of integration. Substituting back and simplifying,

$$H = \frac{\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) - \frac{\hat{a}^\dagger + \hat{a}}{\sqrt{2\omega}}J(t)$$

We will now consider a simplified case where $J(t)$ only acts during the interval $0 < t < T$ and vanishes outside. We will not any assume the exact form of $J(t)$ here but give the schematic form as follows:



Then,

$$\hat{a}(t > T) = \left[\hat{a}(t=0) + \frac{i}{\sqrt{2\omega}}\int_0^T e^{i\omega t'}J(t')dt'\right]e^{-i\omega t}$$
$$= [\hat{a}(t=0) + J_0]e^{-i\omega t}$$
$$\hat{a}^\dagger(t > T) = \left[\hat{a}^\dagger(t=0) - \frac{i}{\sqrt{2\omega}}\int_0^T e^{-i\omega t'}J(t')dt'\right]e^{i\omega t}$$

$$= [\hat{a}^\dagger(t=0) + J_0^*]e^{i\omega t}$$

We find:

$$[\hat{a}(t > T), \hat{a}^\dagger(t > T)] = 1$$

We adopt the following notation, going forward:

$$\hat{a}(t > T) \equiv \hat{a}_{\text{out}} \qquad\qquad \hat{a}(t \leq 0) \equiv \hat{a}_{\text{in}}$$

For $t < 0$:

$$H = \frac{\omega}{2}(2\hat{a}_{\text{in}}^\dagger \hat{a}_{\text{in}} + 1) = \omega\left(\hat{a}_{\text{in}}^\dagger \hat{a}_{\text{in}} + \frac{1}{2}\right)$$

For $t > T$:

$$H = \omega\left(\hat{a}_{\text{in}}^\dagger \hat{a}_{\text{in}} + J_0 \hat{a}_{\text{in}}^\dagger + J_0^* \hat{a}_{\text{in}} + |J_0|^2 + \frac{1}{2}\right) = \omega\left(\hat{a}_{\text{out}}^\dagger \hat{a}_{\text{out}} + \frac{1}{2}\right)$$

We see that the Hamiltonian which was diagonal for $t < 0$ is no longer diagonal in $\hat{a}_{\text{in}}$. This is where the idea of bogoliubov transformation comes in, which finds the new basis defined by new creation and annihilation operator diagonalizing the Hamiltonian. The state which get annihilated by $\hat{a}_{\text{out}}$ are the new vacuum. The old vacuum

$$\hat{a}_{\text{out}} |0\rangle_{\text{in}} = (\hat{a}_{\text{in}} + J_0) |0\rangle_{\text{in}} = J_0 |0\rangle_{\text{in}}$$

From this we understand that the two creation and annihilation operator in two different region define their own vacuum which are distinct from each other. This is also one of the reason why we can not use the standard S matrix formalism as that requires the initial and final vacuum to only differ by a phase factor. Initially,

$${}_{\text{in}}\langle 0| H |0\rangle_{\text{in}} = \frac{\omega}{2}$$

Finally,

$$\begin{aligned} {}_{\text{in}}\langle 0| H |0\rangle_{\text{in}} &= {}_{\text{in}}\langle 0| \omega \left[\frac{1}{2} + \hat{a}_{\text{out}}^\dagger \hat{a}_{\text{out}}\right] |0\rangle_{\text{in}} \\ &= \omega\left[\frac{1}{2} + |J_0|^2\right] \end{aligned}$$

Notice that the $|0\rangle_{\text{in}}$ is no longer the lowest-energy state as $t \geq T$. This occurs when the system's energy varies in response to the external force $J(t)$. Subtracting the zero point energy $\omega/2$, we discover that the initial ground state is now in an excited state with energy $|J_0|^2\omega$. In the absence of this force, $\hat{a}_{\text{in}} = \hat{a}_{\text{out}}$, thus the state $|0\rangle_{\text{in}}$ always describes the vacuum state.

The motivation for considering the forced harmonic oscillator before going into particle creation models in curved spacetime is that we can think of quantum fields as a set of harmonic oscillators. A free field can be treated as a set of infinitely many free harmonic oscillators, whereas in curved spacetime, the fields have to be thought of as a collection of forced harmonic oscillators. This has the implication that initial ground state of harmonic oscillators when interpreted as vacuum state with no particle, turns into excited state due to action of external force is interpreted as state with particles. So the action of time dependent source changes the definition of vacuum. This is the mechanism used in particle production in the presence of time dependent background.

4.2 Particle creation in Bernard and Duncan Model

According to the equivalence principle, the physics for an accelerated observer and an observer in curved spacetime are equivalent. Therefore, it is natural to investigate the same phenomenon in curved spacetime. We already explored one part of that problem in Schwarzschild spacetime. In this chapter, we start with a simpler model where the time-dependent metric is Minkowskian at $t = \pm\infty$ and is varying in between. We will learn that the vacuum state at $t = -\infty$ evolves into a non-vacuum state at $t = +\infty$ similar to what we just saw in the case of forced harmonic oscillator. The metric tensor in FLRW spacetime is given as:

$$ds^2 = dt^2 - a^2(t)dx^2,$$

where the $a(t)$ is the scale factor and it determines the expansion behavior of the universe. It is much easier to perform calculations in the conformal coordinates where all the spacetime axes scale in same manner. To this motivation, we define a new time parameter η, called conformal time, as $d\eta = dt/a(t)$. In the new coordinates, the metric becomes

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2) = C(\eta)(d\eta^2 - dx^2),$$

and $C(\eta) = a^2(\eta)$ is defined to be the new conformal scale factor. For the rest of the part, we make the following choice for the conformal scale factor[14]:

$$C(\eta) = A + B\tanh(\rho\eta),$$

where A, B and ρ are constants. A conformal scale factor is asymptotically Minkowskian in the limit $\eta \to \pm\infty$ since

$$\lim_{\eta\to\pm\infty} C(\eta) = A \pm B.$$

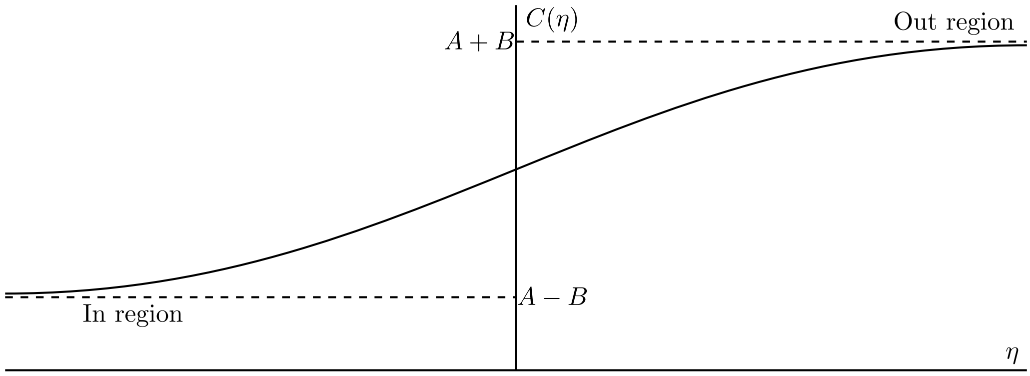


Figure 4.1: A qualitative picture of the evolution of the scale factor in conformal time emerges, depicting a scenario where both the in-region and out-region are flat, but feature **distinct** metric tensors. For particle creation to take place, the vacuum has to transform rather than stay invariant under time translation of metric.

The asymptotic behaviour of the conformal scale factor can be seen in the figure 4.1. The static universe corresponding to $\eta \to -\infty$ is called the in-region, while the static universe corresponding to $\eta \to \infty$ is called the out-region. Even though both the in-region and out-region resemble Minkowski spacetime, however, as we will see, the definition of vacuum in both of them will be different.

Introducing a scalar field

The equation of motion for massive scalar field is given as:

$$(\Box + m^2)\phi(x,\eta) = 0 \tag{4.1}$$

We introduce a complete set of orthonormal mode solutions $u_k(x,\eta)$ of above in conformal coordinates, that obey the properties

$$(u_k, u_l) = \delta_{kl},$$
$$(u_k^*, u_l^*) = -\delta_{kl},$$
$$(u_k, u_l^*) = 0.$$

The mode expansion of scalar field $\phi(x,\eta)$ can be given as:

$$\phi(x,\eta) = \sum_k \left[a_k u_k(x,\eta) + a_k^\dagger u_k^*(x,\eta)\right],$$

where a_k and $a_k^\dagger$ are annihilation and creation operators, respectfully.

Since we have assumed spacetime to be isotropic and homogeneous, a natural separation of mode function into space and time part for the scalar mode functions u_k is given as:

$$u_k(x,\eta) = \frac{1}{\sqrt{2\pi}} e^{ikx}\chi_k(\eta) \tag{4.2}$$

Substituting the mode functions (4.2) into equation of motion (4.1):

$$\frac{d^2}{d\eta^2}\chi_k(\eta) + \left[k^2 + C(\eta)m^2\right]\chi_k(\eta) = 0,$$

This differential equation can be solved in terms of hypergeometric functions, however, our interest is limited to the form of solution in the in-region ($\eta \to -\infty$). The solution is given as[14]:

$$\lim_{\eta\to-\infty} u_k^{\rm in}(x,\eta) = \frac{1}{\sqrt{4\pi\omega_{\rm in}}} e^{ikx - i\omega_{\rm in}\eta}, \quad \omega_{\rm in} = [k^2 + m^2(A-B)]^{1/2}$$

The mode function for the out-region are:

$$\lim_{\eta\to\infty} u_k^{\rm out}(x,\eta) = \frac{1}{\sqrt{4\pi\omega_{\rm out}}} e^{ikx - i\omega_{\rm out}\eta}, \quad \omega_{\rm out} = [k^2 + m^2(A+B)]^{1/2}$$

If the particles were massless, then we would have not observed any particle production. Since, the two mode functions in the different static regions are clearly different for **massive** particles, we will observe particle production. Instead of expressing $\hat{a}_{\rm out}$ in terms of linear combination of $\hat{a}_{\rm in}$ and $\hat{a}_{\rm in}^\dagger$, we will follow a slightly different approach. We will express $u_k^{\rm in}(x,\eta)$ as a linear combination of the real and imaginary part of $u_k^{\rm out}(x,\eta)$:

$$u_k^{\rm in}(x,\eta) = \alpha_k u_k^{\rm out}(x,\eta) + \beta_k u_k^{\rm out *}(x,\eta).$$

To find the Bogoliubov coefficients α_k and β_k, we use the full solution:

$$u_k^{\rm in}(x,\eta) = \frac{1}{\sqrt{4\pi\omega_{\rm in}}} e^{i\{kx - \omega_+\eta - \frac{\omega_-}{\rho}\ln[2\cosh(\rho\eta)]\}} \; {}_2F_1\left(1+\frac{i\omega_-}{\rho}, \frac{i\omega_-}{\rho}, 1-\frac{i\omega_{\rm in}}{\rho}; \frac{1}{2}[1+\tanh(\rho\eta)]\right)$$

$$u_k^{\rm out}(x,\eta) = \frac{1}{\sqrt{4\pi\omega_{\rm out}}} e^{i\{kx - \omega_+\eta - \frac{\omega_-}{\rho}\ln[2\cosh(\rho\eta)]\}} \; {}_2F_1\left(1+\frac{i\omega_-}{\rho}, \frac{i\omega_-}{\rho}, 1+\frac{i\omega_{\rm out}}{\rho}; \frac{1}{2}[1-\tanh(\rho\eta)]\right)$$

where

$$\omega_\pm = \frac{1}{2}(\omega_{\rm out} \pm \omega_{\rm in})$$

We explicitly make use of the following properties of hypergeometric functions[1].

$$\begin{aligned} {}_2F_1(a,b,c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\; {}_2F_1(a,b,1+a+b-c;1-z) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\; {}_2F_1(c-a,c-b,1+c-a-b;1-z) \\ {}_2F_1(a,b,c;z) &= (1-z)^{c-a-b}\; {}_2F_1(c-a,c-b,c;z) \end{aligned}$$

we find

$$\alpha_k = \left(\frac{\omega_{\rm out}}{\omega_{\rm in}}\right)^{1/2} \frac{\Gamma(1-i\omega_{\rm in}/\rho)\Gamma(-i\omega_{\rm out}/\rho)}{\Gamma(-i\omega_+/\rho)\Gamma(1-i\omega_+/\rho)}, \tag{4.3}$$

and

$$\beta_k = \left(\frac{\omega_{\rm out}}{\omega_{\rm in}}\right)^{1/2} \frac{\Gamma(1-i\omega_{\rm in}/\rho)\Gamma(i\omega_{\rm out}/\rho)}{\Gamma(i\omega_-/\rho)\Gamma(1+i\omega_-/\rho)} e^{2ikx} \tag{4.4}$$

Using Eqs. (4.3) and (4.4) and Euler's reflection formula for Gamma function, we find:

$$|\alpha_k|^2 = \frac{\sinh^2(\pi\omega_{\rm out}/\rho)}{\sinh(\pi\omega_{\rm in}/\rho)\sinh(\pi\omega_{\rm out}/\rho)}$$

and

$$|\beta_k|^2 = \frac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_{\rm in}/\rho)\sinh(\pi\omega_{\rm out}/\rho)}.$$

we can check that it satisfies the normalization condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

The mean paricle density in the out mode would be given as:

$${}_{\rm in}\langle 0|\, n_k \,|0\rangle_{\rm in} = |\beta_k|^2$$

This is to be interpreted as following: the $|0\rangle_{\rm in}$ are not populated with in modes but only out-modes. As such, we need detector for out-mode to see particles in the $|0\rangle_{\rm in}$ vacuum.

[1] using $\omega_{\rm out} = 2\omega_\pm \mp \omega_{\rm in}$, and $\omega_{\rm in} + \omega_- = \omega_{\rm in} + \frac{1}{2}(\omega_{\rm out} - \omega_{\rm in}) = \frac{2\omega_{\rm in}+\omega_{\rm out}-\omega_{\rm in}}{2} = \frac{\omega_{\rm out}+\omega_{\rm in}}{2} = \omega_+$

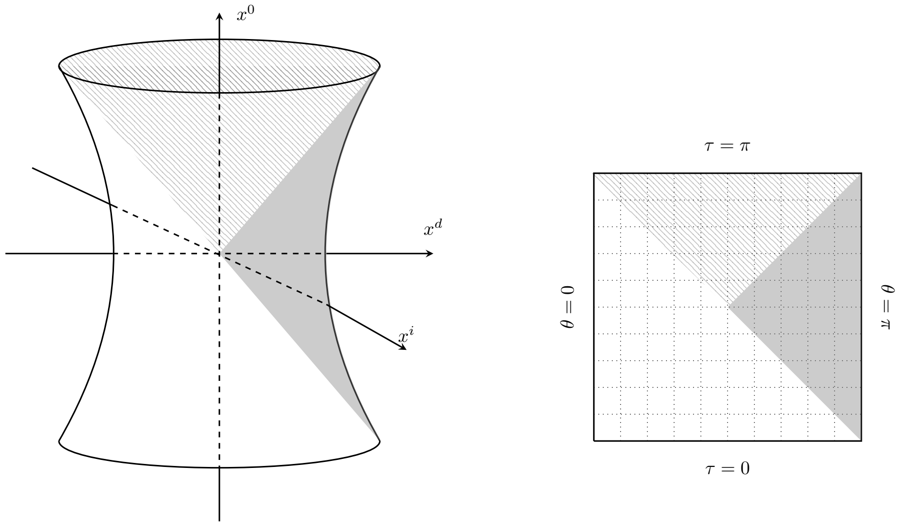


Figure 4.2: The embedding and penrose diagram of de Sitter spacetime. Shaded region depicts region of de Sitter spacetime observable to an observer at south pole $\theta = \pi$. The dark shaded region is the static patch, it is the intersection between the region of space that can affect the observer and the region that can be affected by them.

4.3 Particle Production in de Sitter Space

In this section, we will expand our understanding by exploring the effects of gravity on particle production in de Sitter space. In contrast to the previous section where we studied field theory in a conformally flat FLRW universe, we will now examine the impact of the gravitational field on particle production in de Sitter space. As we discussed earlier, in the context of a massless scalar field in a conformally flat spacetime, there was no particle creation. However, this section will reveal that even massless scalars can experience particle production in de Sitter space due to the presence of the de Sitter horizon. To begin our investigation, we will first consider massless scalar fields.

Massless Scalar field in deSitter space

The metric tensor during inflation is approximately that of de Sitter space ($a(t) \approx e^{Ht}$) in flat slicing is given as:

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 d\vec{x}^2 \\ &= \frac{1}{H^2\eta^2}(-d\eta^2 + d\vec{x}^2) \end{aligned}$$

where we defined conformal time as $d\eta = {}^{dt}/_{a(t)} = -d({}^{e^{-Ht}}/_{H})$. The Lagrangian for classical scalar field is given as:

$$L = \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - V(\phi)\right] \tag{4.5}$$

The equation of motion becomes:

$$\Box\phi + m^2\phi + V'(\phi) = 0$$

for massless and free theory, we have $m = 0$ and $V(\phi) = 0$:

$$\begin{aligned} \Box\phi = \partial_\mu\partial^\mu\phi &= 0 \\ \frac{1}{a}\left(\frac{\partial}{\partial\eta}\frac{1}{a}\right)\frac{\partial\phi}{\partial\eta} + \frac{1}{a^2}\frac{\partial^2\phi}{\partial\eta^2} + \frac{3H}{a}\frac{\partial}{\partial\eta}\phi - \frac{1}{a^2}\nabla^2\phi &= 0 \end{aligned}$$

using $a = -{}^{1}/_{\eta H}$

$$\frac{1}{a^2}\frac{\partial^2\phi}{\partial\eta^2} + \frac{2H}{a}\frac{\partial\phi}{\partial\eta} - \frac{1}{a^2}\nabla^2\phi = 0$$

Thus,

$$\boxed{H^2\eta^2\frac{\partial^2\phi}{\partial\eta^2} - 2H^2\eta\frac{\partial\phi}{\partial\eta} - H^2\eta^2\nabla^2\phi = 0}$$

In momentum space:[2]

$$\ddot{\phi}_{\vec{k}} - \frac{2}{\eta}\dot{\phi}_{\vec{k}} + k^2\phi_{\vec{k}} = 0$$

where $\dot{}$ means derivative with respect to η. It has the solution of the form:[3]

$$\phi_k = c_1\underbrace{\frac{H}{\sqrt{2k^3}}(1-ik\eta)e^{ik\eta}}_{f_k} + c_2\underbrace{\frac{H}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta}}_{\bar{f}_k}$$

with the normalization of mode function, chosen so that

$$\begin{aligned}\bar{f}_k(\partial_t f_k) - (\partial_t\bar{f}_k)f_k &= -H\eta[\bar{f}_k(\partial_\eta f_k) - (\partial_\eta\bar{f}_k)f_k] \\ &= -iH^3\eta^3\end{aligned}$$

with the choice of inner product in curved spacetime given as:

$$\langle f,g\rangle = i\int d^3x\sqrt{|h|}\{f^\star(\partial_t g) - (\partial_t f^\star)g\} = \langle g,f\rangle^\star = -\langle g^\star, f^\star\rangle$$

we get (with the $x-$dependence of mode function coming from $e^{i\vec{k}\cdot\vec{x}}$):

$$\langle f_k, f_{k'}\rangle = i\int\frac{d^3x}{H^3\eta^3}\times -iH^3\eta^3e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3\delta(\vec{k}-\vec{k}')$$

Before we discuss the alternative derivation of the mode function, It'd be better to discuss the time evolution of superhorizon modes characterized by $k << aH$:

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} - \frac{1}{a^2}\nabla^2\phi = 0$$

in momentum space

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} + \frac{1}{a^2}k^2\phi = 0$$

In the limit $k/a << H \implies k\eta << 1 \implies k\eta \to 0$, the mode function becomes constant. However, from EOM we get:

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} = 0 \implies \phi = A\frac{e^{-Ht}}{H} + B$$

We see that modes outside the comoving horizon suffer exponential decay and eventually as $\eta \to 0$ it become constant. The second way to derive the mode function is via redefinition $\phi_{\vec{k}} = -H\eta u_{\vec{k}}$[4]. Then, we get:

$$\begin{aligned}\ddot{\phi}_{\vec{k}} - \frac{2}{\eta}\dot{\phi}_{\vec{k}} + k^2\phi_{\vec{k}} &= 0 \\ -H\frac{\partial}{\partial\eta}\left(u_{\vec{k}} + \eta\dot{u}_{\vec{k}}\right) + \frac{2H}{\eta}\left(u_{\vec{k}} + \eta\dot{u}_{\vec{k}}\right) - k^2H\eta u_{\vec{k}} &= 0 \\ \ddot{u}_{\vec{k}} + \left(k^2 - \frac{2}{\eta^2}\right)u_{\vec{k}} &= 0\end{aligned}$$

[2]we have dropped the $e^{i\vec{k}\cdot\vec{x}}$ factors

[3]for continuous case:

$$\phi(x,\eta) = \int dk[c_k f_k e^{i\vec{k}\cdot\vec{x}} + c.c]$$

for descrete case:

$$\phi(x,\eta) = \sum_k c_k\underbrace{f_k e^{i\vec{k}\cdot\vec{x}}}_{f_k(x)}$$

[4]because we don't know $\dot{\phi}$ at infinite past, so we'd like to get rid of it before we can take the limit

which is much easier to solve. It has the solution given as:

$$u_{\vec{k}} = \frac{1}{\sqrt{2k}}\left(1 \pm \frac{i}{k\eta}\right)e^{\pm ik\eta}$$

In the early time limit $\eta \to -\infty$, which essentially turns the de Sitter Klein Gordon Equation into Minkowski one. We have

$$\begin{aligned} \ddot{u}_{\vec{k}} + \left(k^2 - \cancelto{0}{\frac{2}{\eta^2}}\right)u_{\vec{k}} &= 0 \\ \ddot{u}_{\vec{k}} + k^2 u_{\vec{k}} &= 0 \end{aligned} \tag{4.6}$$

which has the solution

$$u_{\vec{k}} = \frac{1}{\sqrt{2k}}e^{\pm ik\eta}$$

where we have chosen the overall normalization, so that the solutions have Wronskian $\pm i$. So that the mode functions at infinite past are orthonormal i.e. $\langle u_{\vec{k}}, u_{\vec{k}'}\rangle = (2\pi)^3\delta(\vec{k} - \vec{k}')$. At late time ($\eta \to 0$), we can ignore terms independent of η:

$$\begin{aligned} \ddot{u}_{\vec{k}} + \left(k^2 - \frac{2}{\eta^2}\right)u_{\vec{k}} &= 0 \\ \ddot{u}_{\vec{k}} - \frac{2}{\eta^2}u_{\vec{k}} &= 0 \end{aligned}$$

Hence, the solution would be:

$$u_{\vec{k}}(x,\eta) = u_{\vec{k}}(x)\eta^{-1} + \overline{u}_{\vec{k}}(x)\eta^2$$

Therefore, we can express the field at late time as:

$$\phi_{\vec{k}} \approx \sum_{\Delta=0,3} O_\Delta(x)\eta^\Delta \tag{4.7}$$

where

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}$$

for general mass.

Particle Production

In the infinite past we defined the mode function as positive energy solution (minkowski mode)[5], we can express the field in de Sitter space at some finite η as:

$$\phi_k = b_k u_k + b_k^\dagger u_k^\star$$

with $\{b_k, b_k^\dagger\}$, a new set of creation and annihilation operators which defines the new vacuum as $b_k|0\rangle = 0$. The bunch davis mode function is given as:

$$u_k = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}$$

and

$$\frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} = \left[1 - \frac{i}{k\eta} - \frac{1}{2}\left(\frac{1}{k\eta}\right)^2\right]\frac{1}{\sqrt{2k}}e^{-ik\eta} + \frac{1}{2(k\eta)^2}e^{-2ik\eta}\frac{1}{\sqrt{2k}}e^{ik\eta}$$

with

$$\left.\begin{aligned} \alpha &= 1 - \tfrac{i}{k\eta} - \tfrac{1}{2(k\eta)^2} \\ \beta &= \tfrac{1}{2(k\eta)^2}e^{-2ik\eta} \end{aligned}\right\} |\alpha|^2 - |\beta|^2 = 1$$

Assuming, at early time, the total number $n_{\text{minkowski}}$ is

$$n_{\text{minkowski}} = 0$$

However, at later time, the number density, n_{BD}, of massless particles in de Sitter space is given as:

$$n_{BD} = |\beta|^2 = \frac{1}{4k^4\eta^4} \neq 0$$

This is another indication of particle production in expanding spacetime.

[5] which is equivalent to choosing Bunch Davis vacuum

Chapter 5

CMB as particle detector

In particle physics, colliders like the Large Hadron Collider (LHC) allow us to probe the properties of fundamental particles at incredibly high energies. One key aspect of this research is looking for certain peaks which confer the existence of particle. The discipline of collider physics involves going from the direct collider observables to the underlying lagrangian of the theory.

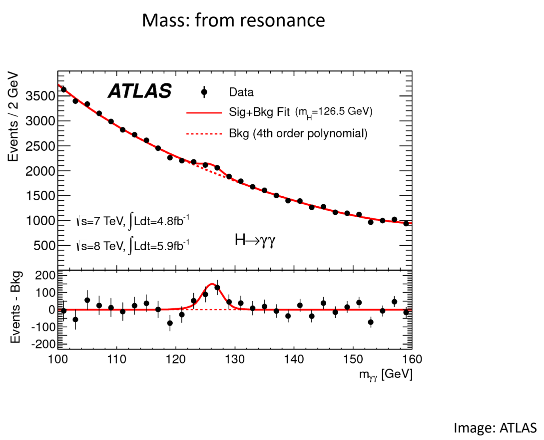


Figure 5.1: The invariant mass distribution of diphoton events recorded by the ATLAS detector, showing a clear excess near 125 GeV. This "bump" represents the signature of a new particle, later confirmed to be the Higgs boson, as predicted by the Standard Model.

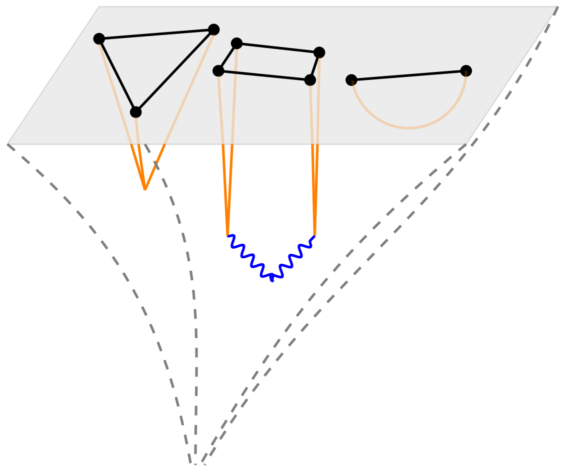


Figure 5.2: Illustration of different types of non-Gaussianity originating from the quantum fluctuation during inflation. The diagram shows the feynman diagrammatic view of how the inflaton perturbation seed the region of overdensity and underdensity. Time flows in the upward direction.

One of the simplest questions one can ask is how to recognize the presence of new particles. In collider Physics, the answer lies in the certain peaks in cross section near the mass scale of intermediate particles. The spin information could be extracted by studying the angular distribution of scattering process. In the cosmological case, instead of peak invariant mass distribution corresponding to the mass of the particle, one finds that the higher-order correlations oscillate when the distance between correlated points varies, and they do so with a frequency given by the mass of the new particle.

In the same spirit, cosmological collider physics proposes to use the cosmic microwave background radiation (CMB) as a "detector" to study non-gaussianity in the early universe, effectively "colliding" new particles that arose from quantum fluctuations during inflation[6]. These particles would not necessarily be ordinary matter but rather new particles that arose from the quantum fluctuations during inflation.

In particular, cosmological collider physics allows us to study the properties of new particles that arise from quantum fluctuations during inflation such as their mass and spin. These particles could have masses comparable to the Hubble scale or even be massless fields that arose during this era. By studying the non-gaussianity present in the primordial fluctuation spectrum, we can effectively reconstruct the history of particle production and decays during this era, gaining insights into the properties and interactions of these particles.

Furthermore, cosmological collider physics provides an alternate approach to test theories of inflation and the early universe. The standard model of cosmology is based on the assumption that the universe underwent an exponential expansion during the era of inflation. However, this expansion could have been driven by a variety of different mechanisms, including the presence of new particles or fields. However by analyzing these

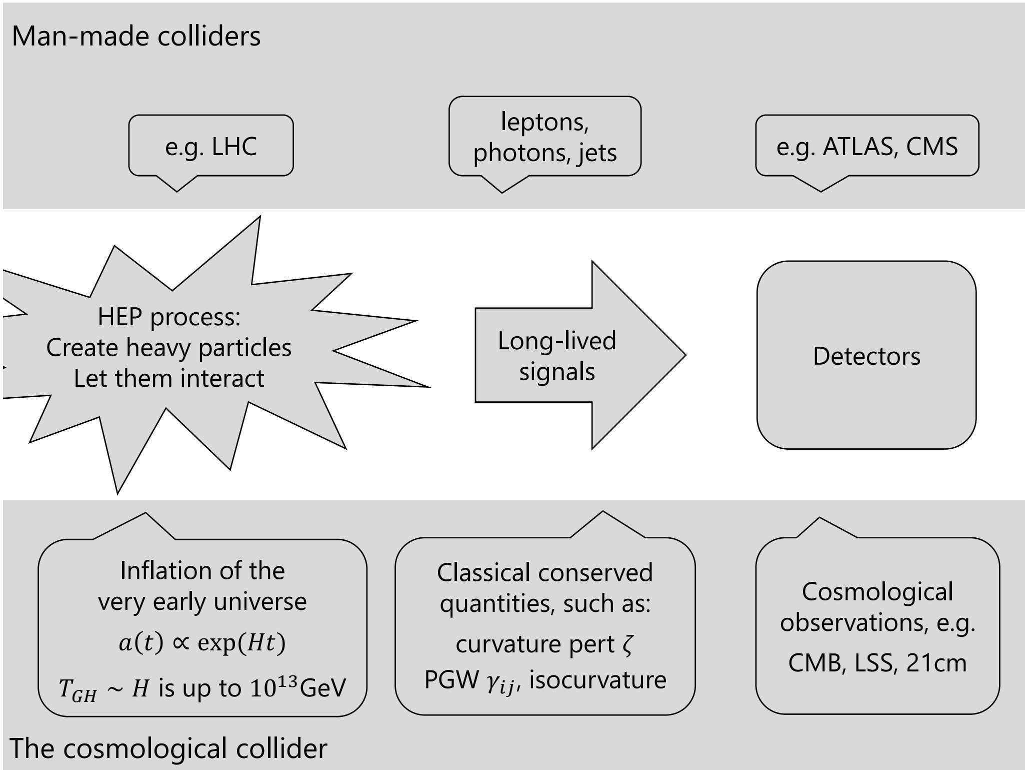


Figure 5.3: A comparision between the cosmological collider and man-made collider to highlight the difference.

statistical signatures left behind in the CMB, we can gain insights into which models are most consistent with observations and which theories predict the correct level of non-gaussianity.

Since, we can not use the S-matrix formalism, we will employ the in-in formalism described in section 2.1.1 for calculating the correlation functions. We assume that Inflation can be modelled by perfect de Sitter spacetime and we ignore any deviation away from it. Therefore, we will now proceed to study the correlation function at the end of Inflation.

5.1 Two Point Function

The two-point correlation function is the cornerstone of cosmological perturbation theory and the primary observable in the study of primordial fluctuations. It captures the statistical distribution of curvature perturbations in the early universe and directly determines the angular power spectrum of the cosmic microwave background (CMB).

In the simplest inflationary models with a single scalar field slowly rolling in a nearly de Sitter background, quantum fluctuations of the field are nearly Gaussian and statistically isotropic. As a result, the full statistical content of these perturbations is encoded in the two-point function alone, with higher-order correlators vanishing in the absence of interactions.
In this section, we compute the two-point function of a scalar field in an inflationary background using the in–in formalism. Starting from the mode expansion of the field and the assumption of the Bunch-Davies vacuum, we derive the standard expression for the power spectrum of inflaton perturbations.

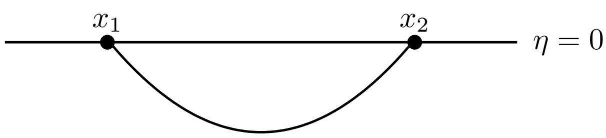


Figure 5.4: in-in Feynman diagrams for equal time two point function on the reheating surface

While the two-point function provides the leading-order statistical prediction of inflation, it is inherently limited in the amount of physical information it can convey. It is insensitive to the presence of interactions or heavy particle content unless those effects modify the vacuum state or the effective mass of the inflaton. For this reason, the study of higher-order correlation functions—such as the bispectrum and trispectrum—becomes essential for probing beyond the minimal inflationary model. But before turning to those, we establish the baseline result: the two-point function and the resulting power spectrum.

The two point correlation function for scalar field in de Sitter spacetime in momentum space at late time (or small η, η' behavior)[1] can be expressed in terms of this mode function as:

$$\langle \phi_{\vec{k}}(\eta)\phi_{\vec{k}'}(\eta') \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}')\ f^\star_{\vec{k}} f_{\vec{k}}$$

The mode function in momentum space is given as:

$$f_k = H\frac{\sqrt{\pi}}{2} e^{-\frac{i}{4}\pi(1+2\nu)} (-\eta)^{3/2} H^{(2)}_\nu(-k\eta) \qquad \text{(where } \nu = \sqrt{\tfrac{9}{4} - \tfrac{m^2}{H^2}}\text{)}$$

if ν is real

$$\begin{aligned}
\langle \phi_{\vec{k}}(\eta)\phi_{-\vec{k}}(\eta') \rangle' &= \frac{H^2 \pi}{4} (\eta\eta')^{3/2} \lim_{k\eta \to 0} \Big[H^{(2)}_\nu(-k\eta) \underbrace{H^{(2)}_\nu(-k\eta')^\star}_{H^{(1)}_\nu(-k\eta')} \Big] \\
&= \frac{H^2 \pi}{4} (\eta\eta')^{3/2} \frac{i}{\pi} \left[\Gamma(\nu) \left(\frac{-k\eta}{2}\right)^{-\nu} + e^{i\pi\nu}\Gamma(-\nu) \left(\frac{-k\eta}{2}\right)^{\nu} \right] \\
&\qquad \times \frac{-i}{\pi} \left[\Gamma(\nu) \left(\frac{-k\eta'}{2}\right)^{-\nu} + e^{i\pi\nu}\Gamma(-\nu) \left(\frac{-k\eta'}{2}\right)^{\nu} \right] \\
&\approx \frac{H^2}{4\pi} (\eta\eta')^{3/2} \left[\Gamma(\nu)^2 \left(\frac{k^2\eta\eta'}{4}\right)^{-\nu} + \Gamma(-\nu)^2 \left(\frac{k^2\eta\eta'}{4}\right)^{\nu} \right] + \text{terms}
\end{aligned}$$

if ν is imaginary

$$\begin{aligned}
\langle \phi_{\vec{k}}(\eta)\phi_{-\vec{k}}(\eta') \rangle' &= \frac{H^2 \pi}{4} (\eta\eta')^{3/2} e^{-i\pi\nu} \lim_{k\eta \to 0} \Bigg[H^{(2)}_\nu(-k\eta) \underbrace{H^{(2)}_\nu(-k\eta')^\star}_{e^{i\pi\nu} H^{(1)}_\nu(-k\eta')} \Bigg] \\
&= \frac{H^2 \pi}{4} (\eta\eta')^{3/2} e^{-i\pi\nu} \frac{i}{\pi} \left[\Gamma(\nu) \left(\frac{-k\eta}{2}\right)^{-\nu} + e^{i\pi\nu}\Gamma(-\nu) \left(\frac{-k\eta}{2}\right)^{\nu} \right] \\
&\qquad \times \frac{-ie^{i\pi\nu}}{\pi} \left[\Gamma(\nu) \left(\frac{-k\eta'}{2}\right)^{-\nu} + e^{-i\pi\nu}\Gamma(-\nu) \left(\frac{-k\eta'}{2}\right)^{\nu} \right] \\
&\approx \frac{H^2}{4\pi} (\eta\eta')^{3/2} \left[\Gamma(\nu)^2 \left(\frac{k^2\eta\eta'}{4}\right)^{-\nu} + \Gamma(-\nu)^2 \left(\frac{k^2\eta\eta'}{4}\right)^{\nu} \right] + \text{terms}
\end{aligned} \tag{5.1}$$

For massive particles $m > \frac{3}{2}H$, we can have imaginary ν, which will show oscillatory behavior at late time signifying particle creation in de Sitter space. The complex conjugate of Hankel function is given as:

$$H^{(2)\star}_{i\mu} = e^{-\pi\mu} H^{(1)}_{i\mu}$$

Then,

$$\begin{aligned}
\lim_{k\eta \to 0} \left| H^{(2)}_\nu(-k\eta) \right|^2 &= e^{-\pi\mu} \lim_{k\eta \to 0} H^{(2)}_\nu(-k\eta) H^{(1)}_\nu(-k\eta) \\
&= \frac{e^{-\pi\mu}}{\pi^2} \left(\Gamma(i\mu)^2 \left(\frac{-k\eta}{2}\right)^{-2i\mu} + \Gamma(-i\mu)^2 \left(\frac{-k\eta}{2}\right)^{2i\mu} - \frac{2\pi \cot(\pi\mu)}{i\mu} \right),
\end{aligned}$$

and the late-time limit of the **power spectrum** is

$$\begin{aligned}
\lim_{k\eta \to 0} P_\phi(\eta, k) &= \lim_{k\eta \to 0} \langle \phi_{\vec{k}}(\eta)\phi_{-\vec{k}}(\eta') \rangle' = \lim_{k\eta \to 0} \left| f_{\vec{k}}(\eta) \right|^2 \\
&= \frac{\pi}{4} e^{\pi\mu} H^2 (-\eta)^3 \lim_{k\eta \to 0} \left| H^{(2)}_\nu(-k\eta) \right|^2 \\
&= \frac{1}{4\pi} H^2 (-\eta)^3 \left(\Gamma(i\mu)^2 \left(\frac{-k\eta}{2}\right)^{-2i\mu} + \Gamma(-i\mu)^2 \left(\frac{-k\eta}{2}\right)^{2i\mu} + \frac{2\pi \coth(\pi\mu)}{\mu} \right).
\end{aligned}$$

[1] It is these late-time statistics that are relevant for our Universe. The fluctuations generated during inflation that survive until late times are the things that serve as the initial conditions for the evolution of the universe at the onset of the hot Big Bang.

Writing $\Gamma(\pm i\mu) \equiv |\Gamma(\pm i\mu)|\, e^{\pm i\delta}$, and using

$$|\Gamma(\pm i\mu)|^2 = \frac{\pi}{\mu \sinh(\pi\mu)},$$

we can write this as

$$\begin{aligned}\lim_{k\eta\to 0} P_\phi(\eta, k) &= \frac{H^2(-\eta)^3}{4\mu\sinh(\pi\mu)}\left[e^{i\delta(\mu)}\left(\frac{-k\eta}{2}\right)^{-2i\mu} + e^{-i\delta(\mu)}\left(\frac{-k\eta}{2}\right)^{2i\mu} + 2\cosh(\pi\mu)\right] \\ &= \frac{H^2(-\eta)^3}{2\mu\sinh(\pi\mu)}\left[\cos\left(2\mu\log\left(\frac{-k\eta}{2}\right) - \delta\right) + \cosh(\pi\mu)\right].\end{aligned}$$

For large masses, $\mu \gg 1$, the power spectrum scales as $1/\sinh(\pi\mu) \to e^{-\pi\mu}$, corresponding to the Boltzmann suppression of the spontaneous production of massive particles. It is also interesting to consider the correlator in the limit $|\eta| << |\eta'|$ and for $|\vec{x} - \vec{x}'| = 0$:

$$\langle\phi(\eta, \vec{x})\phi(\eta', \vec{0})\rangle' = H^2 \frac{\Gamma\left(\frac{3}{2} + i\mu\right)\Gamma\left(\frac{3}{2} - i\mu\right)}{(4\pi)^2}\ {}_2F_1\left(\frac{3}{2} + i\mu, \frac{3}{2} - i\mu, 2; 1 - \frac{-(\eta-\eta')^2 + \vec{x}^2}{4\eta\eta'}\right)$$

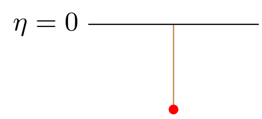
Taking the limit:

$$\begin{aligned}&= H^2 \frac{\Gamma\left(\frac{3}{2} + i\mu\right)\Gamma\left(\frac{3}{2} - i\mu\right)}{(4\pi)^2}\left[\left(\frac{4\eta\eta'}{\eta'^2}\right)^{\frac{3}{2}+i\mu}\frac{\Gamma(2)\Gamma(-2i\mu)}{\Gamma\left(\frac{3}{2} - i\mu\right)\Gamma\left(\frac{1}{2} - i\mu\right)} + c.c.\right] \\ &= \frac{H^2}{4\pi^{5/2}}\left[\left(\frac{\eta}{\eta'}\right)^{\frac{3}{2}+i\mu}\Gamma(-i\mu)\Gamma\left(\frac{3}{2} + i\mu\right) + \left(\frac{\eta}{\eta'}\right)^{\frac{3}{2}-i\mu}\Gamma(i\mu)\Gamma\left(\frac{3}{2} - i\mu\right)\right] + \cdots\end{aligned}$$

if we used the $i\epsilon$ prescription inside hypergeometric function, then we would have ended with:

$$\langle\phi(\eta, 0)\phi(\eta', \vec{0})\rangle' = \frac{H^2}{4\pi^{5/2}}\left[\left(\frac{\eta}{\eta'}e^{-i\pi}\right)^{\frac{3}{2}+i\mu}\Gamma(-i\mu)\Gamma\left(\frac{3}{2} + i\mu\right) + \left(\frac{\eta}{\eta'}e^{-i\pi}\right)^{\frac{3}{2}-i\mu}\Gamma(i\mu)\Gamma\left(\frac{3}{2} - i\mu\right)\right] + \cdots$$

This extra $e^{i\pi}$ gives factors of $e^{\pm\pi\mu}$, which imply that the first term in above is now not suppressed exponentially for large μ while the second term is suppressed by $e^{-2\pi\mu}$. The first term is the one expected where we create a particle at η_0 and destroy it at $\eta > \eta_0$.



5.2 Three Point function

We used the in-in formalism to calculate the two-point function in the preceding section. In this part, we will follow an alternate approach to calculate the three-point function purely from a symmetry perspective. Since this approach is highly constraining, we will use it to calculate the three-point function at late times, where two of the fields are scalar fields, and the third is a tensor field. Our goal is to construct an invariant that respects the constraints imposed by the late time isometries of de Sitter spacetime, which correspond to conformal symmetry. We use Embedding Space formalism for deriving the three point function of two scalar and one vector field respecting conformal symmetry. On the null cone we will have

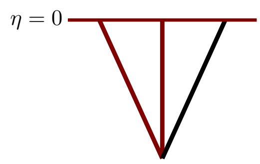


Figure 5.5: In-in feynman diagram of three point function with two scalar field and one vector field or massive field.

$$\langle\phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle = \frac{W_M}{(-2X_1\cdot X_2)^{\alpha_{123}}(-2X_1\cdot X_3)^{\alpha_{132}}(-2X_2\cdot X_3)^{\alpha_{231}}}$$

where the powers α_{ijk} of the scalar factor are determined by the dilatation as in case of scalar operators and the tensor structure W_M equals to

$$W_M = \frac{(-2X_2\cdot X_3)X_{1M} - (-2X_1\cdot X_3)X_{2M} - (-2X_1\cdot X_2)X_{3M}}{(-2X_1\cdot X_2)^{\frac{1}{2}}(-2X_1\cdot X_3)^{\frac{1}{2}}(-2X_2\cdot X_3)^{\frac{1}{2}}}.$$

which satisfies:

$$(X_1)^M W_M = 0$$
$$(X_2)^M W_M = 0$$

$$(X_3)^M W_M = 0$$

The relative sign between terms in the expression for W_M is fixed by transversality and the scaling behavior of correlation function under dilatation is completely determined in the scalar part so the tensor structure have scaling 0 in all variables ($X \to \lambda X \implies W_\mu \to \lambda^0 W_\mu$). Projecting to physical space as:

$$\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle = \frac{\partial X_3^M}{\partial x_3^\mu}\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle$$

we find, as explicitly computed before,

$$\frac{\partial X_3^M}{\partial x_3^\mu}X_{iM} = (x_i - x_3)_\mu, \quad i = 1,2$$
$$-2X_i \cdot X_j = (x_i - x_j)^2, \quad i = 1,2,3 \ (i < j),$$

so that we end up with the tensor structure

$$W_\mu = \frac{|x_2 - x_3|^2(x_1 - x_3)_\mu - |x_1 - x_3|^2(x_2 - x_3)_\mu}{|x_1 - x_2||x_1 - x_3||x_2 - x_3|} = \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}$$

Therefore, the three-point function of two scalars and one spin 1 operators in physical space is given by

$$\begin{aligned}\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle &= \frac{\frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}||x_{13}||x_{23}|}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \\ &= \frac{|x_{23}|^2(x_{13})_\mu - |x_{13}|^2(x_{23})_\mu}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2+1}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1+1}}\end{aligned}$$

Where Δ_1, Δ_2, and Δ_3 are the conformal scaling dimension of ϕ_1, ϕ_2 and J_μ. The three-point function of higher-spin operators $J_{\mu_1\ldots\mu_\ell}$ is constructed from the above, analogously as what we did for the two-point functions, since it turns out that W_μ is the only indexed object for three points that is conformal invariant. The reason why we were able to do this is because the conformal symmetry is so contraining that it only leaves few possibilities for the correlation function upto some constant. This constant is determined by the dynamics of the conformal theory.

5.3 Three Point Function of two conformally coupled fields with a general mass scalar field

In quantum field theory with conformal symmetry, we have two kinds of scalar field. One with mass term which breaks the conformal invariance and another, a massless scalar field with coupling to gravity given as $\mathcal{L}_{\text{int}} = \xi \mathcal{R}\phi^2$ respects it. It is natural to wonder about the interaction between them as one respects the symmetry and another breaks it. We set the conformal scaling dimension $\Delta_1 = \Delta_2 = 2$ and $\Delta_3 = \Delta$, and define the following variables

$$q = \frac{k_1 - k_2}{k_3} \qquad\qquad p = \frac{k_1 + k_2}{k_3}$$

which gives us following

$$\frac{\partial}{\partial k_1} = \frac{\partial p}{\partial k_1}\frac{\partial}{\partial p} = \frac{1}{k_3}\frac{\partial}{\partial p}; \qquad \frac{\partial}{\partial k_2} = \frac{\partial p}{\partial k_2}\frac{\partial}{\partial p} = \frac{1}{k_3}\frac{\partial}{\partial p}; \qquad \frac{\partial}{\partial k_3} = \frac{\partial p}{\partial k_3}\frac{\partial}{\partial p} = \frac{k_1 + k_2}{-k_3^2} = -\frac{p^2}{k_1 + k_2}\frac{\partial}{\partial p}$$

also

$$\frac{\partial^2}{\partial k_1^2} = \left(\frac{1}{k_3}\right)^2\frac{\partial^2}{\partial p^2}; \quad \frac{\partial^2}{\partial k_2^2} = \left(\frac{1}{k_3}\right)^2\frac{\partial^2}{\partial p^2}; \qquad \frac{\partial^2}{\partial k_3^2} = -\frac{p^2}{k_1 + k_2}\frac{\partial}{\partial p}\left(-\frac{p^2}{k_1 + k_2}\frac{\partial}{\partial p}\right) = \frac{1}{k_3^2}\left(2p\frac{\partial}{\partial p} + p^2\frac{\partial^2}{\partial p^2}\right)$$

Next, we will solve the Conformal Ward Identity (CWI) for the conformally coupled scalar fields with the following ansatz:

$$\begin{aligned}\langle \phi(\vec{k}_1)\phi(\vec{k}_2)\sigma_\Delta(\vec{k}_3)\rangle &= k_3^{\Delta_1+\Delta_2+\Delta-2d}G(q,p) \\ &= k_3^{\Delta-2}G(q,p) \qquad \text{(where } d = 3)\end{aligned}$$

Using ward identity for special conformal transformation. The generator for SCT takes the following representation in momentum space:

$$\hat{K}_\mu = -\left[2(\Delta - d)\frac{\partial}{\partial k^\mu} - 2k^\alpha \frac{\partial^2}{\partial k^\mu \partial k^\alpha} + k_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha}\right]$$
$$= k_\mu \left[-2(\Delta - d + 1)\frac{1}{k}\frac{\partial}{\partial k} + \frac{\partial^2}{\partial k^2}\right]$$

Since, in our case $d = 3$, the CWI takes the form of:

$$\vec{b}\cdot\vec{k}\left[-2(\Delta - 2)\frac{1}{k}\frac{\partial}{\partial k} + \frac{\partial^2}{\partial k^2}\right]\langle\phi(\vec{k}_1)\phi(\vec{k}_2)\sigma_\Delta(\vec{k}_3)\rangle$$

If we assume $\vec{b}\cdot\vec{k}_3 = 0$, then it leads to

$$(\vec{b}\cdot\vec{K}_1 + \vec{b}\cdot\vec{K}_2)k_3^{\Delta-2}G = 0$$
$$\left(\vec{b}\cdot\vec{k}_1\frac{\partial^2}{\partial k_1^2} - \vec{b}\cdot\vec{k}_1\frac{\partial^2}{\partial k_3^2}\right)k_3^{\Delta-2}G = 0$$
$$k_3^{\Delta-2}\vec{b}\cdot\vec{k}_1(\partial_{k_1}^2 - \partial_{k_2}^2)G = 0 \implies \partial_q\partial_p G = 0$$

Assuming, $G(q,p) \equiv G(p)$

$$\left[\vec{b}\cdot\vec{K}_1 + \vec{b}\cdot\vec{K}_2 + \vec{b}\cdot\vec{K}_3\right]k_3^{\Delta-2}G = 0$$
$$\left[\underbrace{(\vec{b}\cdot\vec{k}_1 + \vec{b}\cdot\vec{k}_2)}_{=-\vec{b}\cdot\vec{k}_3}\frac{1}{k_3^2}\partial_p^2 + \vec{b}\cdot\vec{k}_3\left\{\frac{(\Delta-2)(1-\Delta)}{k_3^2} + \frac{2p\partial_p + p^2\partial_p^2}{k_3^2}\right\}\right]k_3^{\Delta-2}G = 0$$
$$k_3^{\Delta-2}[(p^2-1)\partial_p^2 + 2p\partial_p + (\Delta-2)(1-\Delta)]G = 0$$

where we used

$$\left(-2(\Delta-3)\frac{1}{k}\frac{\partial}{\partial k_3} + \frac{\partial^2}{\partial k_3^2}\right)k_3^{\Delta-2}G = \frac{(\Delta-2)(1-\Delta)}{k_3^2}k_3^{\Delta-2}G + \frac{k_3^{\Delta-2}}{k_3^2}\left(2p\frac{\partial G}{\partial p} + p^2\frac{\partial^2 G}{\partial p^2}\right)$$

The final expression

$$[(p^2-1)\partial_p^2 + 2p\partial_p + (\Delta-2)(1-\Delta)]G = 0 \tag{5.2}$$

is hypergeometric differential equation with poles at $p = \pm 1$ and the solution is given as:

$$G(p) = {}_2F_1\left(2-\Delta, \Delta-1, 1; \frac{1-p}{2}\right)$$

We note that the bispectrum has branch point at $p = 1, \infty$. This result will be important in upcoming part.

5.4 Four Point Function in ϕ^3 and ϕ^4 Theories

In contrast to earlier scenario, where we could calculate three point function purely from symmetry considerations, the four-point functions are no longer uniquely determined by transformation properties alone. This is because when you have at least four distinct points, you can construct conformally invariant cross-ratios[15]. These cross-ratios are special functions of the points that remain unchanged under all conformal transformations. For a configuration of four points, we find that there exist exactly two independent cross-ratios.

$$\frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|}; \qquad \frac{|x_{12}||x_{34}|}{|x_{23}||x_{14}|}$$

As they are not uniquely determined by symmetry constraints, we will therefore use in-in formalism for deriving the results.

$\lambda\phi^4$ type interaction

We begin by stating the Wightman propagator,

$$W_k(\eta,\eta') = \frac{H^2}{2k}(\eta\eta')e^{-ik(\eta-\eta')}$$

Then, the four point function at later time η_* is given by:

$$
\begin{aligned}
I_+ &= -i\lambda \int_{-\infty}^{\eta_*} D\eta W_{k_1}(\eta_*, \eta) W_{k_2}(\eta_*, \eta) W_{k_3}(\eta_*, \eta) W_{k_4}(\eta_*, \eta) \\
&= -i \frac{\lambda (H\eta_*)^4}{16 k_1 k_2 k_3 k_4} \int_{-\infty}^{\eta_*} d\eta\, e^{i(k_1+k_2+k_3+k_4)\eta} \\
&= -\frac{\lambda (H\eta_*)^4}{16 k_1 k_2 k_3 k_4 (k_1+k_2+k_3+k_4)}.
\end{aligned}
$$

This contribution is again purely real ($I_+ = I_-$), so the full correlator is just twice the above answer:

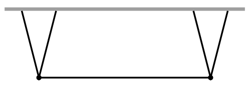
$$
\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \rangle' = -\frac{\lambda (H\eta_*)^4}{8 k_1 k_2 k_3 k_4} \frac{1}{k_t}
$$

$\lambda\phi^3$ type interaction

We begin by considering a conformally coupled scalar. We can write the Wightman and Feynman propagators explicitly

$$
\begin{aligned}
W_k(\eta, \eta') &= H^2 \eta \eta' \frac{1}{2k} e^{-ik(\eta - \eta')}, \\
G_F(k; \eta, \eta') &= H^2 \eta \eta' \underbrace{\frac{1}{2k} \left[e^{-ik(\eta-\eta')} \theta(\eta - \eta') + e^{ik(\eta - \eta')} \theta(\eta' - \eta) \right]}_{G_F^{(\text{flat})}}
\end{aligned}
$$

Notice that these are exactly $H^2 \eta \eta'$ times their flat-space counterparts. (This is another manifestation of the conformal invariance of the conformally coupled scalar.) The I_{++} contribution is[2]

$I_{++} \equiv$ 

$$
\begin{aligned}
&= (-i\lambda)^2 \int_{-\infty}^{\eta_*} D\eta' D\eta'' W_{k_1}(\eta_*, \eta') W_{k_2}(\eta_*, \eta') G_F(k_I; \eta', \eta'') W_{k_3}(\eta_*, \eta'') W_{k_4}(\eta_*, \eta'') \\
&= -\frac{\lambda^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta'}{\eta'} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta''}{\eta''} e^{ik_{12}\eta'} G_F^{(\text{flat})}(k_I; \eta', \eta'') e^{ik_{34}\eta''},
\end{aligned}
$$

where, we expressed the second line in terms of the flat-space Feynman propagator. This integral is a bit tricky to evaluate because it is divergent. As we will see, the divergences in I_{--} and I_{++} cancel against similar divergences in I_{-+} and I_{+-}. In order to isolate the divergent piece, it is therefore useful to add and subtract a contribution to the above integral as

$$
I_{++} = -\frac{\lambda^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta'}{\eta'} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta''}{\eta''} e^{ik_{12}\eta'} \Bigg[\underbrace{G_F^{(\text{flat})}(k_I; \eta', \eta'') - \frac{1}{2k_I} e^{ik_I(\eta' + \eta'')}}_{G_B^{(\text{flat})}} + \frac{1}{2k_I} e^{ik_I(\eta' + \eta'')} \Bigg] e^{ik_{34}\eta''}
$$

where we have defined

$$
G_B^{(\text{flat})}(k_I; \eta', \eta'') = G_F^{(\text{flat})}(k_I; \eta', \eta'') - \frac{1}{2k_I} e^{ik_I(\eta' + \eta'')}
$$

We can now perform the integral involving G_B via a trick. We note that

$$
-i \int_a^\infty \mathrm{d}x e^{ix\eta} = \frac{1}{\eta} e^{ia\eta},
$$

So we can write

$$
\begin{aligned}
I_{++} = \frac{\lambda^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4} \Bigg[& \underbrace{\int_{k_{12}}^\infty \mathrm{d}x \int_{k_{34}}^\infty \mathrm{d}y \int_{-\infty}^{\eta_*} \mathrm{d}\eta' \mathrm{d}\eta'' e^{ix\eta'} e^{iy\eta''} G_B^{(\text{flat})}(k_I; \eta', \eta'')}_{\text{connected part}} \\
& \underbrace{- \frac{1}{2k_I} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta'}{\eta'} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta''}{\eta'''} e^{i(k_{12}+s)\eta'} e^{i(k_{34}+s)\eta''}}_{\text{disconnected part}} \Bigg]
\end{aligned}
$$

[2]vertices and propagators below the conformal boundary are time ordered and above the conformal boundar are anti-time ordered.

The time integrals in the first line now only involve flat-space quantities and can therefore easily be computed

$$I(x,y) \equiv \int_{-\infty}^{0} \mathrm{d}\eta' \mathrm{d}\eta'' e^{ix\eta'} e^{iy\eta''} G_B^{(\mathrm{flat})}(k_I;\eta',\eta'') = -\frac{1}{(x+y)(x+k_I)(y+k_I)}$$

The integral was performed using

$$\int_{-\infty}^{\eta'} d\eta'' \int_{-\infty}^{0} d\eta' e^{ix\eta'} e^{iy\eta''} e^{-ik_I(\eta'-\eta'')} = -\frac{1}{(k_I+y)(x+y)}$$
$$\int_{-\infty}^{0} d\eta'' \int_{-\infty}^{\eta''} d\eta' e^{ix\eta'} e^{iy\eta''} e^{ik_I(\eta'-\eta'')} = -\frac{1}{(k_I+x)(x+y)}$$
$$\int_{-\infty}^{0} d\eta'' \int_{-\infty}^{0} d\eta' e^{ix\eta'} e^{iy\eta''} e^{ik_I(\eta'+\eta'')} = -\frac{1}{(k_I+x)(k_I+y)}$$

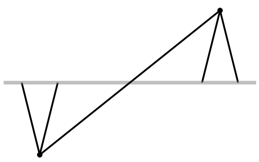
We split I_{++} into two pieces: a "connected" part, $I_{++}^{(c)}$, which is the energy integral of $I(x,y)$, and a "disconnected" part, $I_{++}^{(d)}$, which involves integral of terms without time ordering. Performing the integrals of the connected piece, we get:

$$\begin{aligned} I_{++}^{(c)} &\equiv \frac{\lambda^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \; I(x,y) \\ &= \frac{\lambda^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \left[\mathrm{Li}_2\left(\frac{E-E_L}{E}\right) + \mathrm{Li}_2\left(\frac{E-E_R}{E}\right) + \log\left(\frac{E_L}{E}\right) \log\left(\frac{E_R}{E}\right) - \frac{\pi^2}{6} \right] \end{aligned}$$

where Li_2 is the dilogarithm and we define the variables $E_L \equiv k_{12} + k_I$ (the energy flowing into the left vertex), $E_R \equiv k_{34} + k_I$ (same for the right vertex) and the total energy $E \equiv k_{12} + k_{34} = k_t$. We now calculate the disconnected part:

$$\begin{aligned} I_{++}^{(d)} &\equiv -\frac{\lambda^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta'}{\eta'} e^{i(k_{12}+s)\eta'} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta''}{\eta''} e^{i(k_{34}+s)\eta''} \\ &= -\frac{\lambda^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \log(iE_L\eta_*) \log(iE_R\eta_*) \end{aligned} \tag{5.3}$$

This contribution diverges in the limit $\eta_* \to 0$, which is why we split it off from $I_{++}^{(c)}$. These divergences cancel with the $I_{\mp\pm}$ pieces, so everything is finite. Using the Feynman rules, we find

$I_{+-} \equiv$  $= \lambda^2 \int_{-\infty}^{\eta_*} D\eta' D\eta'' W_{k_1}(\eta_*,\eta') W_{k_2}(\eta_*,\eta') W_{k_I}(\eta'',\eta') W_{k_3}(\eta'',\eta_*) W_{k_4}(\eta'',\eta_*)$

$$\begin{aligned} &= \frac{\lambda^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta'}{\eta'} e^{i(k_{12}+k_I)\eta'} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta''}{\eta''} e^{-i(k_{34}+k_I)\eta''} \\ &= \frac{\lambda^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \log(iE_L\eta_*) \log(-iE_R\eta_*) \end{aligned} \tag{5.4}$$

In the last line, we evaluated the integrals, noting that the different $i\epsilon$ prescriptions required for convergence of the integrals lead to different signs of the arguments of the log for the η' and η''integrals. Adding (5.3) to (5.4)we get,

$$I_{++}^{(d)} + I_{+-} = \frac{\lambda^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \left(-\pi \log(iE_L\eta_*)\right) \tag{5.5}$$

The sum of the $I_{--}^{(d)}$ and I_{-+} contributions gives the complex conjugate of (5.5). Putting everything together, we therefore find

$$I_{++}^{(d)} + I_{+-} + I_{--}^{(d)} + I_{-+} = \frac{\lambda^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \pi^2 \tag{5.6}$$

Adding it all, the full correlator is[3]

$$\langle \phi_{\vec{k}_1} \phi_{\vec{k}_2} \phi_{\vec{k}_3} \phi_{\vec{k}_4} \rangle' = 2^2 (I_{++} + I_{+-} + c.c.)$$

[3]The reason for symmetry factor is absense of 3! in $\mathcal{L}_{\mathrm{int}}$ and because we have fixed the intermediate particle being exchanged.

$$
\begin{aligned}
&= 2^2\frac{\lambda^2H^2\eta_*^4}{16k_1k_2k_3k_4k_I}\left[\mathrm{Li}_2\left(\frac{E-E_L}{E}\right)+\mathrm{Li}_2\left(\frac{E-E_R}{E}\right)+\log\left(\frac{E_L}{E}\right)\log\left(\frac{E_R}{E}\right)+\frac{\pi^2}{3}\right]\\
&= \frac{\lambda^2H^2\eta_*^4}{4k_1k_2k_3k_4k_I}\left[\mathrm{Li}_2\left(\frac{k_{12}-k_I}{k_t}\right)+\mathrm{Li}_2\left(\frac{k_{34}-k_I}{k_t}\right)+\log\left(\frac{k_{12}+k_I}{k_t}\right)\log\left(\frac{k_{34}+k_I}{k_t}\right)+\frac{\pi^2}{3}\right]
\end{aligned} \tag{5.7}
$$

When $k_I \to 0$, we have

$$
\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4}\rangle' \approx \frac{\lambda^2H^2\eta_*^4}{8k_1k_2k_3k_4}\left[\frac{\pi^2}{k_I}-2\frac{k_I}{k_{12}k_{34}}+\cdots\right]
$$

We are interested in these terms because they give rise to interesting power law behavior in position space. In position space, there are further contribution to the Operator Product Expansion limit.

$$
\begin{aligned}
\langle\varphi(\vec{x}_1)\cdots\varphi(\vec{x}_4)\rangle &\propto \frac{1}{x_{13}^2x_{24}^2}\frac{1}{(z-\bar{z})}\left[2\mathrm{Li}_2(z)-2\mathrm{Li}_2(\bar{z})+\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}\right]\\
\text{with}\quad z\bar{z} &= \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad (1-z)(1-\bar{z})=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}.
\end{aligned}
$$

We find that in the limit $x_{12}\to 0$, we have $\ln(z\bar{z})\to\infty$. These singularities are less interesting for our purposes because they do not signal the existence of new particles. They simply reflect interactions between the particles that already exist. Such terms can arise both from a contact interaction or from analytic terms in $\vec{k}_I$ in the exchange diagrams. However, we will see that when the intermediate particles are massive, it leads to a unique signal in the CMB, an analogue of resonance in cosmological collider physics.

5.5 Four point function with an intermediate massive field

We can now consider conformally coupled scalars φ coupled to a general massive scalar field σ with an interaction vertex $\int \lambda\varphi^2\sigma$. The late time expectation value is given by the following expression:

$$
\begin{aligned}
\langle\varphi_{\vec{k}_1}(\eta_0)\cdots\varphi_{\vec{k}_4}(\eta_0)\rangle' &= \frac{\eta_0^4 2^2\lambda^2}{16k_1k_2k_3k_4}I_E(k_{12},k_{34},k_I)+\text{two other diagrams}\\
I_E(k_{12},k_{34},k_I) &\equiv I_{++}+I_{+-}+I_{-+}+I_{--}\\
I_{\pm\pm} &= (\pm i)(\pm i)\int_{-\infty}^{0}\frac{d\eta}{\eta^2}e^{\pm ik_{12}\eta}\int_{-\infty}^{\eta}\frac{d\eta'}{\eta'^2}e^{\pm ik_{34}\eta'}\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{\pm\pm}
\end{aligned} \tag{5.8}
$$

where the $\pm$ signs indicate the type of branch in the integration contour, and consequently the sign in subscript of $\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{\pm\pm}$ represents the propagator along that branch. The equations of motion for $\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{\pm\pm}$ are given as:

$$
\left(\eta^2\frac{\partial^2}{\partial\eta^2}-2\eta\frac{\partial}{\partial\eta}+k_I^2\eta^2+\frac{m^2}{H^2}\right)\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{++} = -iH^2\eta^2\eta'^2\delta(\eta-\eta') \tag{5.9}
$$

Now, consider the following auxiliary object:

$$
\tilde{F} = -\int_{-\infty}^{0}\frac{d\eta}{\eta^2}\frac{d\eta'}{\eta'^2}e^{ik_{12}\eta}e^{ik_{34}\eta'}\left(\eta^2\frac{\partial^2}{\partial\eta^2}-2\eta\frac{\partial}{\partial\eta}+k_I^2\eta^2+\frac{m^2}{H^2}\right)\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{++} \tag{5.10}
$$

and evaluate it in two ways. First, we use (5.9) to write

$$
\tilde{F} = iH^2\int_{-\infty}^{0}d\eta e^{i(k_{12}+k_{34})\eta} = \frac{H^2}{k_{12}+k_{34}},
$$

Next, we integrate the time derivatives by parts[4] in (5.10) and trade factors of η for $-i\partial_{k_{12}}$ to obtain

$$
\begin{aligned}
\tilde{F} &= -\int_{-\infty}^{0}\frac{d\eta}{\eta^2}\frac{d\eta'}{\eta'^2}\left[-(k_{12}^2-k_I^2)\eta^2+2ik_{12}\eta+\frac{m^2}{H^2}-2\right]e^{ik_{12}\eta}e^{ik_{34}\eta'}\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{++}\\
&= -\left[(k_{12}^2-k_I^2)\partial_{k_{12}}^2+2k_{12}\partial_{k_{12}}+\frac{m^2}{H^2}-2\right]\int_{-\infty}^{0}\frac{d\eta}{\eta^2}\frac{d\eta'}{\eta'^2}e^{ik_{12}\eta}e^{ik_{34}\eta'}\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{++}
\end{aligned}
$$

[4] we are dropping the boundary term at $\eta = 0$.

using (5.8)

$$\left[(k_{12}^2-k_I^2)\partial_{k_{12}}^2+2k_{12}\partial_{k_{12}}+\frac{m^2}{H^2}-2\right]I_{++}=\frac{H^2}{k_{12}+k_{34}}$$

under the redefinition:

$$p=\frac{k_{12}}{k_I} \qquad p'=\frac{k_{34}}{k_I} \qquad I_{\pm\pm}=\frac{G_{\pm\pm}(p,p')}{k_I}$$

hence,

$$k_I^2\left[(p^2-1)\partial_p^2+2p\partial_p+\frac{m^2}{H^2}-2\right]I_{++}=H^2\frac{k_I}{p+p'} \tag{5.11}$$

If we used

$$\left(\eta'^2\frac{\partial^2}{\partial\eta'^2}-2\eta'\frac{\partial}{\partial\eta'}+k_I^2\eta'^2+\frac{m^2}{H^2}\right)\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{++}=-iH^2\eta^2\eta'^2\delta(\eta-\eta')$$

then by redoing all the same steps we arrive at:

$$k_I^2\left[(p'^2-1)\partial_{p'}^2+2p'\partial_{p'}+\frac{m^2}{H^2}-2\right]I_{++}=H^2\frac{k_I}{p+p'} \tag{5.12}$$

from (5.11) and (5.12), we have:

$$\left[(p^2-1)\partial_p^2+2p\partial_p-(p'^2-1)\partial_{p'}^2-2p'\partial_{p'}\right]I_{++}=0$$

Similarly, we can use

$$\left(\eta'^2\frac{\partial^2}{\partial\eta'}-2\eta'\frac{\partial}{\partial\eta'}+k_I^2\eta'^2+\frac{m^2}{H^2}\right)\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{+-}=0$$

to show that

$$\left[(k_{12}^2-k_I^2)\partial_{k_{12}}^2+2k_{12}\partial_{k_{12}}+\frac{m^2}{H^2}-2\right]I_{+-}=0.$$

If we used

$$\left(\eta'^2\frac{\partial^2}{\partial\eta'^2}-2\eta'\frac{\partial}{\partial\eta'}+k_I^2\eta'^2+\frac{m^2}{H^2}\right)\langle\sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{++}=0$$

in (5.10) then, we could show:

$$\left[(k_{34}^2-k_I^2)\partial_{k_{34}}^2+2k_{34}\partial_{k_{34}}+\frac{m^2}{H^2}-2\right]I_{+-}=0.$$

let's define the following new coordinate

$$p=\frac{k_{12}}{k_I} \qquad p'=\frac{k_{34}}{k_I} \qquad I_{\pm\pm}=\frac{G_{\pm\pm}(p,p')}{k_I}$$

and then,

$$\left[(p^2-1)\partial_p^2+2p\partial_p+\frac{m^2}{H^2}-2\right]G_{+-}=0$$
$$\left[(p'^2-1)\partial_{p'}^2+2p'\partial_{p'}+\frac{m^2}{H^2}-2\right]G_{+-}=0.$$

Notice that "time" has completely disappeared from the problem. The differential equation has been formulated in terms of boundary momenta with possible singularities at $p=\pm1$ and $p=\infty$. We demand that the solution is regular at $p=1$ and we will impose the leading singular behavior at $p=-1$ is properly normalized. The above equation is very similar to (5.2), therefore, we expect the four point function I_{+-} to factorize as product of two three point functions which is regular at $p,p'=1$.

$$\begin{aligned}G_{+-}&\propto\langle OOX\rangle\langle OOX\rangle\\&=\frac{\pi^2}{2\cosh^2(\pi\mu)}\,{}_2F_1\left(\frac12+i\mu,\frac12-i\mu,1,\frac{1-p}{2}\right){}_2F_1\left(\frac12+i\mu,\frac12-i\mu,1,\frac{1-p'}{2}\right)\end{aligned} \tag{5.13}$$

where the proportionality constant is determined by imposing the normalization condition at $p = -1$:

$$\begin{aligned} I_{+-} &= \lim_{p,p' \to -1} \int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{ik_{12}\eta} \int_{-\infty}^{\eta} \frac{d\eta'}{\eta'^2} e^{-ik_{34}\eta'} \frac{\eta\eta'}{2k_I} e^{ik_I(\eta-\eta')} \\ &= \frac{1}{2k_I} \ln(1+p)\ln(1+p') \implies G_{+-} = \frac{1}{2}\ln(1+p)\ln(1+p') \end{aligned}$$

We know the following from

$$\begin{aligned} \Gamma\left(\frac{1}{2} + i\mu\right)\Gamma\left(1 - \frac{1}{2} - i\mu\right) &= \frac{\pi}{\sin\left[\pi\left(\frac{1}{2} + i\mu\right)\right]} \\ &= -\frac{\pi}{\cos(i\pi\mu)} \end{aligned}$$

we get:

$$\begin{aligned} \lim_{p\to -1} {}_2F_1\left(\frac{1}{2} + i\mu, \frac{1}{2} - i\mu, 1, \frac{1-p}{2}\right) &\to -\frac{\ln\left(1 - \frac{1-p}{2}\right)}{\Gamma\left(\frac{1}{2} + i\mu\right)\Gamma\left(\frac{1}{2} - i\mu\right)} \\ &\approx \frac{\cosh(\pi\mu)}{\pi}\ln(1+p) \end{aligned}$$

We quickly observer that the four point correlation function has branch point at $p = -1$ and $p = \infty$.

Another way to evaluate the same is to use the long-distance approximation discussed in Equation (5.1). This long-distance behavior is independent of the $\pm$ subindices as Feynman Propagator and Wightmann Propagator coincide. This is basically because we are probing separations where the two vertices are outside each others' lightcones, so that the time-ordering does not matter.

$$\begin{aligned} \langle \sigma_{k_I}(\eta)\sigma_{-k_I}(\eta')\rangle'_{\pm\pm} &\approx \frac{H^2}{4\pi}(\eta\eta')^{3/2}\left[\Gamma(\nu)^2\left(\frac{k_I^2\eta\eta'}{4}\right)^{-\nu} + \Gamma(-\nu)^2\left(\frac{k_I^2\eta\eta'}{4}\right)^{\nu}\right] \\ &= (\eta\eta')^{3/2+\nu}\frac{H^2}{4\pi}\left[\Gamma(-\nu)^2\left(\frac{k_I^2}{4}\right)^{\nu}\right] + (\eta\eta')^{3/2-\nu}\frac{H^2}{4\pi}\left[\Gamma(\nu)^2\left(\frac{k_I^2}{4}\right)^{-\nu}\right] \\ &= (\eta\eta')^{\Delta}\frac{H^2}{4\pi}\left[\Gamma(-i\mu)^2\left(\frac{k_I^2}{4}\right)^{i\mu}\right] + \underbrace{(\eta\eta')^{\tilde{\Delta}}\frac{H^2}{4\pi}\left[\Gamma(i\mu)^2\left(\frac{k_I^2}{4}\right)^{-i\mu}\right]}_{c.c.} \end{aligned}$$

using $\nu = i\mu$ and defining a new variable $J_\pm$:

$$\begin{aligned} J_\pm(k_{12}) &= \pm i\int_{-\infty}^{0} \frac{d\eta}{\eta^2} e^{\pm ik_{12}\eta}(-\eta)^{\Delta} \\ &= -(\mp i)^{\Delta}(k_{12})^{1-\Delta}\Gamma(\Delta - 1) \\ &= -(\mp e^{i\frac{\pi}{2}})^{\Delta}(k_{12})^{1-\Delta}\Gamma(\Delta - 1) \end{aligned}$$

using $\Delta = \frac{3}{2} + i\mu$, we get from (5.8):

$$\begin{aligned} I_{++} + I_{+-} + I_{-+} + I_{--} = \Bigg(& J_+(k_{12})J_+(k_{34}) + J_+(k_{12})J_-(k_{34}) + J_-(k_{12})J_+(k_{34}) \\ & + J_-(k_{12})J_-(k_{34})\Bigg)\frac{H^2}{4\pi}\left[\Gamma(-i\mu)^2\left(\frac{k_I^2}{4}\right)^{i\mu}\right] + (\mu \leftrightarrow -\mu) \end{aligned}$$

setting $H = 1$

$$= [J_+(k_{12}) + J_-(k_{12})][J_+(k_{34}) + J_-(k_{34})]\frac{1}{4\pi}\left[\Gamma(-i\mu)^2\left(\frac{k_I^2}{4}\right)^{i\mu}\right] + (\mu \leftrightarrow -\mu)$$

$$I_E = \frac{1}{2\pi\sqrt{k_{12}k_{34}}}\left[\left(\frac{k_I^2}{4k_{12}k_{34}}\right)^{i\mu}(1 + i\sinh\pi\mu)\Gamma(-i\mu)^2\Gamma\left(\frac{1}{2} + i\mu\right)^2 + c.c.\right]$$

Hence

$$\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4}\rangle'_{k_I\to 0} \sim \frac{\eta^4\lambda^2}{4k_1k_2k_3k_4} I_E$$

we see that the signal oscillates with a frequency and phase given by the mass of the new particles. This is the analog of resonances in collider physics. We can see that the overall amplitude scales as $e^{-\pi\mu}$ for large μ by using the following:

$$
\begin{aligned}
\Gamma(\pm i\mu)^2\Gamma\left(\frac{1}{2}\mp i\mu\right)^2 &= |\Gamma(\pm i\mu)|^2\left|\Gamma\left(\frac{1}{2}\mp i\mu\right)\right|^2 e^{\pm i\delta} \\
&= \frac{\pi}{\mu\sinh(\pi\mu)}\frac{\pi}{\cosh(\pi\mu)}e^{\pm i\ \delta}
\end{aligned}
$$

We can simplify the I_E for large mass:

$$
\begin{aligned}
I_E &= \frac{1}{2\pi\sqrt{k_{12}k_{34}}}\frac{\pi^2}{\mu\sinh(\pi\mu)\cosh(\pi\mu)}\left[\left(\frac{k_I^2}{4k_{12}k_{34}}\right)^{i\mu}(1+i\sinh(\pi\mu))e^{i\delta}+c.c\right] \\
&\sim \frac{\pi e^{-\pi\mu}}{2\mu\sqrt{k_{12}k_{34}}}\sin\left[\mu\ln\left(\frac{k_I^2}{4k_{12}k_3}\right)+\delta\right]
\end{aligned}
$$

Notice that above contains oscillations in the logarithm of the ratio $k_I^2/k_{12}k_{34}$. It means we have a bump in the squeezed limit as compared to bump near the mass scale in energy distribution. If the leading $e^{-\pi\mu}$ contribution comes from either I_{++} or I_{--}. Then, subleading $e^{-2\pi\mu}$ contribution comes from I_{+-} or I_{-+}, which can be seen as follows:

$$
\begin{aligned}
I_{++} &= J_+(k_{12})J_+(k_{34})\frac{H^2}{4\pi}\left[\Gamma(-i\mu)^2\left(\frac{k_I^2}{4}\right)^{i\mu}\right]+(\mu\leftrightarrow-\mu) \\
&= ie^{\pi\mu}\frac{H^2}{4\pi\sqrt{k_{12}k_{34}}}\left(\frac{k_I^2}{4k_{12}k_{34}}\right)^{i\mu}\Gamma(-i\mu)^2\Gamma\left(\frac{1}{2}+i\mu\right)^2+(\mu\leftrightarrow-\mu) \\
I_{-+} &= J_-(k_{12})J_+(k_{34})\frac{H^2}{4\pi}\left[\Gamma(-i\mu)^2\left(\frac{k_I^2}{4}\right)^{i\mu}\right]+(\mu\leftrightarrow-\mu) \\
&= \frac{H^2}{4\pi\sqrt{k_{12}k_{34}}}\left(\frac{k_I^2}{4k_{12}k_{34}}\right)^{i\mu}\Gamma(-i\mu)^2\Gamma\left(\frac{1}{2}+i\mu\right)^2+(\mu\leftrightarrow-\mu)
\end{aligned}
$$

It satisfies the analytic continuation condition:

$$
I_{+\pm}(k_{12},k_{34},k_I) = -I_{-\pm}(e^{i\pi}k_{12},k_{34},k_I)
$$

We can derive the same using (5.13) by taking the limit $p,p'\to\infty$.

$$
\lim_{|z|\to\infty}{}_2F_1(a,b,c;z)\approx\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}
$$

and duplication formula

$$
\begin{aligned}
\Gamma(2i\mu) &= \frac{2^{2i\mu}}{2\sqrt{\pi}}\Gamma(i\mu)\Gamma\left(\frac{1}{2}+i\mu\right) \\
\Gamma(-2i\mu) &= \frac{2^{-2i\mu}}{2\sqrt{\pi}}\Gamma(-i\mu)\Gamma\left(\frac{1}{2}-i\mu\right) \\
\left|\Gamma\left(\frac{1}{2}+i\mu\right)\right|^2 &= \frac{\pi}{\cosh(\pi\mu)}
\end{aligned}
$$

Then

$$
G_{+-}\sim\frac{1}{2\sqrt{\pi p}}\left[(2p)^{-i\mu}\Gamma(-i\mu)\Gamma\left(\frac{1}{2}+i\mu\right)+c.c\right]\times\frac{1}{2\sqrt{\pi p'}}\left[(2p')^{-i\mu}\Gamma(-i\mu)\Gamma\left(\frac{1}{2}+i\mu\right)+c.c\right]
$$

upon simplification, we get

$$
\begin{aligned}
G_{+-} &\sim \frac{k_I}{4\pi(k_{12}k_{34})^{\frac{1}{2}}}\left[(4pp')^{i\mu}\Gamma(i\mu)^2\Gamma\left(\frac{1}{2}-i\mu\right)^2+c.c.\right]+\dots \\
I_{+-} &\sim \frac{1}{4\pi(k_{12}k_{34})^{\frac{1}{2}}}\left[(4pp')^{i\mu}\Gamma(i\mu)^2\Gamma\left(\frac{1}{2}-i\mu\right)^2+c.c.\right]+\underbrace{\text{terms independent of }k_I}_{\text{analytic}}
\end{aligned}
$$

In the small k_I or $p, p' \to \infty$ limit, we can obtain the results for the other integrals by performing analytic continuation and then adding them up we get back the same expression of I_E keeping only the non-analytic terms in k_I. We see that $p^{i\mu}$ has branch point at $p = 0$ and $p = -1, \infty$ from $\log(1 + p)$, therefore, the branch cut lies in the interval $p \in [-1, 0]$. Since we can move the branch cut by using a different branch of multivalued function but we can not move the branch point. The main argument for interpreting it as a signal for particle production is the presence of branch cut and particle production begins as we approach the branch point in the long wavelength limit.

5.6 Interpretation

The main result of this chapter can be expressed in a more convenient form by separating the result into gaussian part + non-gaussian part via wavefunction formalism, leading to an interesting interference effect on the cosmological scale. For the four point function, we naturally have zeroth order disconnected diagram and second order connected diagram describing interaction.

Figure 5.6: The first diagram represents the disconnected diagram where the fields had gaussian evolution while the second is connected diagram describing creation of massive intermediate particles which decays eventually leading to curvature perturbation on the CMB.

However, we can not attribute the physical process to any one feynman diagram[16]. The probability is given by the square of sums – and not by the sum of squares. The additional "interference term" can not be assigned to a single Feynman diagram. Recall the double slit in QM: observable pattern $=|\text{slit}_1 + \text{slit}_2|^2$. A single Feynman diagram is like the probability amplitude for the electron passing to one slit. If either the disconnected gaussian feynman diagram was describing the physical process or the connected non-gaussian feynman diagram was describing it. There would have been no problem at all. However, the physical process being described is in superposition of all feynman diagrams describing the process. The leading boltzmann supression is being attributed to the interference effect between different diagrams. The probability will be given by the square of sum of amplitude for each feynman diagram. Hence, the leading contribution scaling as $e^{-\pi\mu}$ will arise from the cross term describing gaussian and non-gaussian evolution. Such cross term is to be interpreted as describing interference between producing the pair and not producing the pair of particles. While the subleading term will arise from pair production going as $e^{-2\pi\mu}$. Since, the probability for creating the particle goes as $e^{-2\pi\mu}$. We will write the wavefunction for the process as:

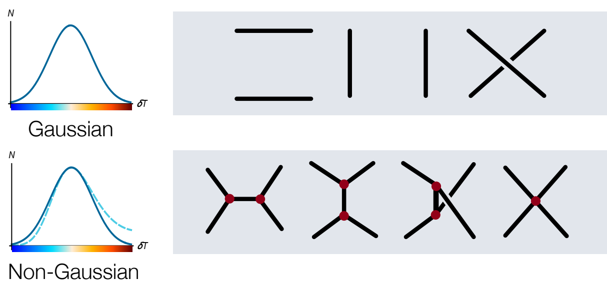


Figure 5.7: Zeroth order disconnected diagrams describing gaussian evolution of fields and second order connected diagrams describing non-gaussian evolution.

$$\psi = \underbrace{\psi_{\text{no pair}}}_{\substack{\text{gaussian evolution} \\ \mathcal{L}_{\text{int}}=0}} + e^{-\pi\mu}\psi_{\text{pair}}$$

we assume that, the non-gaussianity here is originating purely from pair production.

This effect is very quantum mechanical since we see the oscillations of the wavefunction of a quantum particle. Here we are seeing the oscillations because we have the interference between producing the pair and not producing the pair of particles.

$$|\psi|^2 \approx |\psi_{\text{no pair}}|^2 + e^{-\pi\mu}\psi_{\text{no pair}}\psi^\star_{\text{pair}} + e^{-\pi\mu}\psi_{\text{pair}}\psi^\star_{\text{no pair}}$$

In flat space quantum field theory, we know that either the connected diagrams dominate or the disconnected diagrams dominate. However, what we observe here is that, for the non-Gaussian part, the interference term dominates over connected diagram. What it implies is that looking at the sky we can never tell where the particle production took place, we can only detect the deviation in the probability distribution.

Chapter 6

Observing Inflationary Particle Physics in the CMB

The cosmic microwave background (CMB) serves as a powerful window into the physics of the early universe. The primordial density fluctuations imprinted on the CMB anisotropies carry signatures of quantum processes that occurred during inflation. In particular, the non-Gaussian features of the CMB—encoded in higher-point correlation functions—offer a unique opportunity to study particle production and interactions at energy scales far beyond the reach of terrestrial colliders.

In this chapter, we discuss how such signatures can be extracted from CMB data, focusing on the role of the bispectrum and trispectrum as tools to detect non-Gaussianities. We explain how heavy particles coupled to the inflaton can leave characteristic imprints—such as oscillatory "clock signals"—in the momentum dependence of these correlation functions. We also highlight the observational challenges and prospects associated with measuring these signals in current and future CMB experiments, thereby establishing a bridge between high-energy particle physics and precision cosmology.

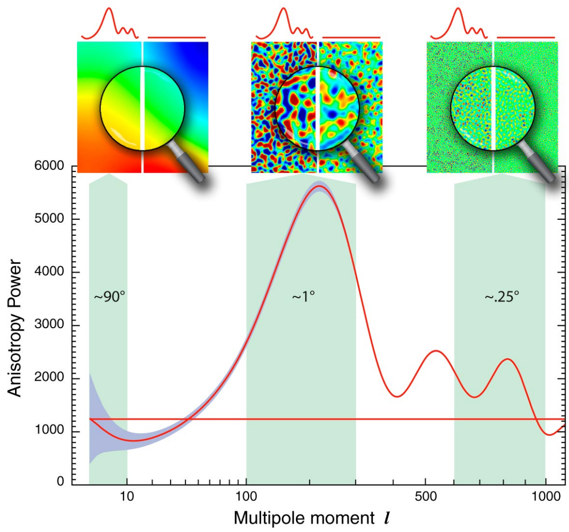


Figure 6.1: The illustration of power spectrum and the type of fluctuation each range of multipole moment l encodes within the spectrum.

6.1 Statistical Structure of Gaussian Fields

To understand the statistical properties of primordial fluctuations and their imprints in the cosmic microwave background (CMB), it is crucial to begin with the simplest case: a single random variable x drawn from a probability distribution. If x follows a Gaussian distribution with zero mean, then its probability density function (PDF) is given by:

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

This distribution is normalized such that:

$$\int_{-\infty}^{\infty} P(x)\, dx = 1$$

Here, σ^2 denotes the variance of the distribution, and it fully characterizes the statistics of x. We can now compute the moments of x to reveal its statistical structure:

$$\begin{aligned} \langle x \rangle &= 0 && \text{(mean)} \\ \langle x^2 \rangle &= \sigma^2 && \text{(variance)} \\ \langle x^3 \rangle &= 0 && \text{(skewness vanishes)} \\ \langle x^4 \rangle &= 3\sigma^4 && \text{(kurtosis)} \end{aligned}$$

$$\langle x^5 \rangle = 0 \qquad \text{(odd moments vanish)}$$
$$\vdots$$

In general, for a Gaussian variable, all odd moments vanish, and even moments are entirely determined by σ. This reflects the high degree of symmetry and simplicity in the Gaussian distribution: all statistical information is encoded in the two-point function (the variance in this case).

6.2 Multivariate Gaussian Fields and Correlation Functions

Extending to the case of a spatially distributed field, we consider a set of random variables $\{x_1, x_2, \ldots, x_N\}$, which may represent field values at different spatial points. These variables can obey a multivariate Gaussian distribution, defined by the joint PDF[17]:

$$P(x_1, x_2, \ldots, x_N) = \frac{1}{(2\pi)^{N/2}|\xi|^{1/2}} \exp\left(-\frac{1}{2}\sum_{i,j} x_i(\xi^{-1})_{ij}x_j\right)$$

Here, $\xi_{ij} = \langle x_i x_j \rangle$ defines the *covariance matrix*, also referred to as the *two-point correlation function.* The normalization condition ensures:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(x_1, \ldots, x_N)\, dx_1 \ldots dx_N = 1$$

The expectation values or moments are:

$$\begin{aligned}
\langle x_i \rangle &= 0 \\
\langle x_i x_j \rangle &= \xi_{ij} \\
\langle x_i x_j x_k \rangle &= 0 \\
\langle x_i x_j x_k x_l \rangle &= \xi_{ij}\xi_{kl} + \xi_{ik}\xi_{jl} + \xi_{il}\xi_{jk} \\
\langle x_i x_j x_k x_l x_m \rangle &= 0 \\
&\vdots
\end{aligned}$$

The structure of the four-point and higher even-order moments is dictated by **Wick's theorem**, which states that all higher-order moments of a Gaussian field can be expressed as products of two-point functions. For example:

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle \langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle \langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle \langle x_2 x_3 \rangle$$

In the special case of a single Gaussian variable:

$$\langle x^4 \rangle = 3\langle x^2 \rangle^2 = 3\sigma^4$$

In cosmology, these statistical properties play a central role. The primordial fluctuations generated during inflation are expected to be nearly Gaussian, with the two-point function—the power spectrum—capturing their dominant behavior. Deviations from Gaussianity (i.e., non-zero higher-point correlations beyond what Wick's theorem predicts) provide crucial clues about interactions and particle production during inflation.

The covariance matrix ξ_{ij}, or equivalently the two-point function $\langle \delta(\vec{x}_i)\delta(\vec{x}_j) \rangle$, is the key observable in the CMB temperature and polarization maps. By extending our study to the bispectrum and trispectrum, we can test for and constrain possible new physics beyond the minimal inflationary paradigm.

6.2.1 Homogeneity and Isotropy

Let $\vec{q}$ be spatial coordinates. Then, random variables at a given spatial position will be given by $X(\vec{q})$. The 2-point correlation function between 2 points $\vec{q}_i$ and $\vec{q}_j$ is

$$\xi_{ij} = \langle X(\vec{q}_i) X(\vec{q}_j) \rangle$$

Let $\vec{r}_{ij}$ be a vector connecting two points.

$$\xi_{ij} = \langle X(\vec{q}_i) X(\vec{q}_i + \vec{r}_{ij}) \rangle$$

Thus, ξ_{ij} would depend on $\vec{q}_i$ as well as $\vec{r}_{ij}$.

- Homogeneity (translational invariance) : statistical homogeneity demands ξ_{ij} not depend on $\vec{q}_i$

$$\therefore \xi_{ij} \neq \xi(\vec{q}_i, \vec{r}_{ij}), \quad \text{but} \quad \xi_{ij} := \xi(\vec{r}_{ij})$$

- Isotropy (rotational invariance): when seen from a given point $\vec{q}_i$, ξ_{ij} does not depend on the direction of $\vec{r}_{ij}$.

$$\therefore \xi_{ij} := \xi(\vec{q}_i, |\vec{r}_{ij}|)$$

- Homogeneity and Isotropy: If ξ_{ij} is isotropic from any point in space, then it must be also homogeneous. We believe that we live in a universe which is isotropic and homogeneous. (But we must keep testing this hypothesis - it's fundamental enough for it to be tested repeatedly!)

$$\boxed{\xi_{ij} \equiv \xi_{ij}(|\vec{r}_{ij}|)}$$

Let us consider a Fourier transform of $x(\vec{q})$,

$$\tilde{X}(\vec{k}) \equiv \int d^3q e^{-i\vec{k}\cdot\vec{q}} X(\vec{q})$$

and consider the covariance matrix given by

$$C_{ij} = \langle \tilde{X}(\vec{k}_i) \tilde{X}^*(\vec{k}_j) \rangle$$

The translational invariance demands

$$C_{ij} = (2\pi)^3 \delta_D^{(3)}(\vec{k}_i - \vec{k}_j) \underbrace{P(\vec{k}_j)}_{\text{power spectrum}}$$

The rotational invariance further demands

$$P(\vec{k}_j) \to P(|\vec{k}_j|)$$

Proof:

$$\begin{aligned} C_{ij} &= \int d^3q \int d^3q' \, e^{-i\vec{k}_i\cdot\vec{q}} e^{i\vec{k}_j\cdot\vec{q}'} \langle X(\vec{q}) X(\vec{q}') \rangle \\ &= \int d^3q \int d^3r e^{-i(\vec{k}_i - \vec{k}_j)\cdot\vec{q}} e^{i\vec{k}_j\cdot\vec{r}} \underbrace{\langle X(\vec{q}) X(\vec{q}+\vec{r}) \rangle}_{\xi(\vec{r})} \end{aligned}$$

Translational Invariance:

$$\begin{aligned} \langle X(\vec{q}) X(\vec{q}+\vec{r}) \rangle &= \xi(\vec{r}) \\ C_{ij} &= \underbrace{\int d^3r e^{i\vec{k}_j\cdot\vec{r}} \xi(\vec{r})}_{P(\vec{k}_j)} \underbrace{\int d^3q e^{-i(\vec{k}_i - \vec{k}_j)\cdot\vec{q}}}_{(2\pi)^3\delta^{(3)}(\vec{k}_i - \vec{k}_j)} \\ &= (2\pi)^3 \delta^{(3)}(\vec{k}_i - \vec{k}_j) P(\vec{k}_j) \end{aligned}$$

Rotational Invariance:

$$\xi(\vec{r}) = \xi(|r|) \implies P(\vec{k}_j) = P(|\vec{k}_j|)$$

Therefore,

$$C_{ij} = (2\pi)^3 \delta^{(3)}(\vec{k}_i - \vec{k}_j) P(|\vec{k}_j|)$$

We can write down the PDF of $\tilde{X}(k)$ as well:

$$P(\tilde{X}(\vec{k}_1), \cdots, \tilde{X}(\vec{k}_N)) = \frac{1}{(2\pi)^{N/2}|C|^{1/2}} e^{-\frac{1}{2}\sum_{ij} \tilde{X}(\vec{k}_i)(C^{-1})_{ij}\tilde{X}^*(\vec{k}_j)}$$

For a translationally and rotationally invariant x, we have

$$P(\tilde{X}(\vec{k}_1), \cdots, \tilde{X}(\vec{k}_N)) = \frac{1}{(2\pi)^{N/2} \left(\prod_i P(|\vec{k}_i|)\right)^{1/2}} e^{-\frac{1}{2}\sum_i \frac{|\tilde{x}(\vec{k}_i)|^2}{P(|\vec{k}_i|)}}$$

which is much simplified.

For this reason, we often deal with the Fourier quantities such as the power spectrum, $P(|\vec{k}|)$, when we analyze the cosmological data sets, as we believe that we live in a statistically homogeneous and isotropic universe.

Bispectrum

We can expand the above argument, and define the "bispectrum," given by

$$\langle \tilde{X}(\vec{k}_1)\tilde{X}(\vec{k}_2)\tilde{X}(\vec{k}_3)\rangle \quad (= 0 \text{ for a Gaussian } \tilde{X})$$

Translational and rotational invariance then demand that this quantity be

$$\langle \tilde{X}(\vec{k}_1)\tilde{X}(\vec{k}_2)\tilde{X}(\vec{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B(|\vec{k}_1|,|\vec{k}_2|,|\vec{k}_3|)$$

6.2.2 Non-Gaussian Statistics

Gaussian statistics is particularly appealing because the probability distribution function (PDF) is completely specified by the two-point correlation function, both in real space and in Fourier space. However, in many physical situations—including the early universe—it becomes essential to consider deviations from Gaussianity.

A natural question arises: *What do we do when the fluctuations are non-Gaussian?*

In general, there is no unique or optimal method to handle an unknown PDF. However, in the context of cosmology, we are guided by observational data: measurements of the CMB suggest that the primordial fluctuations are nearly Gaussian. This means that any deviations from Gaussian statistics must be small and can therefore be treated perturbatively.

A reasonable approach in this regime is to approximate the non-Gaussian PDF as a small deformation of the Gaussian one. That is, we expand around the Gaussian PDF to capture leading-order non-Gaussian features.

Recap: The Taylor expansion of a smooth function $f(x)$ around $x = 0$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=0}$$

We apply this same logic to perturb the Gaussian PDF to model weakly non-Gaussian statistics.

Let $P(x)$ be the true (non-Gaussian) PDF of the field. We expand it around the Gaussian PDF, $P_G(x)$, as:

$$P(x) = P_G(x)\left[1 + \sum_{n=3}^{\infty} \frac{1}{n!}\kappa_n H_n\left(\frac{x}{\sigma}\right)\right]$$

where:

- $P_G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ is the Gaussian PDF,
- $H_n(x)$ are the Hermite polynomials,
- κ_n are the cumulants beyond the second order (e.g., skewness κ_3, kurtosis κ_4, etc.).

This form ensures that deviations from Gaussianity are encoded in higher-order cumulants. The Hermite polynomials serve as an orthogonal basis with respect to the Gaussian measure, which makes them a natural choice for this expansion.

In cosmology, this expansion has a direct interpretation:

- The presence of a non-zero **bispectrum** (three-point function) corresponds to a non-zero skewness, or κ_3.
- The presence of a **trispectrum** (four-point function) corresponds to non-zero kurtosis, or κ_4.

Thus, measurements of higher-order correlation functions in the CMB can be seen as an attempt to reconstruct the full probability distribution of primordial fluctuations, moving beyond the Gaussian approximation.

This framework—perturbing around a Gaussian—is at the heart of modern non-Gaussian cosmology and forms the basis for the field known as *cosmological collider physics*, which seeks to interpret these higher-point statistics in terms of underlying particle physics during inflation.

6.2.3 Gram-Charlier Expansion

Consider a Gaussian PDF within a unit variance $\sigma^2 = 1$ and zero mean. Let's call it $G(x)$:

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Then, suppose that we have a PDF of a weakly non-gaussian random variable, and we want to describe this new PDF, $P(x)$, as a perturbation to $G(x)$. We obtain:

$$\begin{aligned} P(x) &= \sum_{n=0}^{\infty} C_n \frac{\mathrm{d}^n G(x)}{\mathrm{d}x^n} \\ &= G_0(x)[C_0 + (-1)C_1 x + C_2(x^2 - 1) + (-1)C_3(x^3 - 3x) + \dots] \end{aligned}$$

This is so called "Gram-Charlier Expansion". There is another way to express this result and for that we need to define Chebyshev-Hermite Polynomials as

$$\begin{aligned} He_n(x) &\equiv (-1)^n \frac{1}{G(x)} \frac{\mathrm{d}^n G}{\mathrm{d}x^n} \\ &= (-1)^n e^{x^2/2} \frac{\mathrm{d}^n e^{-x^2/2}}{\mathrm{d}x^n} \end{aligned}$$

we have:

$$\begin{aligned} He_0(x) &= 1 \\ He_1(x) &= x \\ He_2(x) &= x^2 - 1 \\ He_3(x) &= x^3 - 3x \\ He_4(x) &= x^4 - 6x^2 + 3 \end{aligned} \tag{6.1}$$

The PDF can be written as:

$$P(x) = G(x) \left[\sum_{n=0}^{\infty} C_n (-1)^n He_n(x) \right] \tag{6.2}$$

In other words, we are expanding the ratio, $P(x)/G(x)$, in terms of the Chebyshev-Hermite polynomials times $(-1)^n$.

$$\frac{P(x)}{G(x)} = \sum_{n=0}^{\infty} C_n \ (-1)^n He_n(x)$$

A nice property of $He_n(x)$ is that it satisfies the following relation:

$$\int_{-\infty}^{\infty} G(x) He_n(x) He_m(x) dx = m! \delta_{mn}$$

which allows us to systematically derive the coefficient, C_n. First, multiply both sides of (6.2) by $He_n(x)$, and integrate over x.

$$\begin{aligned} \int_{-\infty}^{\infty} dx P(x) He_n(x) &= \sum_{m=0}^{\infty} C_m (-1)^m \int_{-\infty}^{\infty} dx G(x) He_m(x) He_n(x) \\ &= \sum_{m=0}^{\infty} C_m (-1)^m m! \delta_{mn} \\ &= (-1)^n n! C_n \end{aligned}$$

$$\boxed{\therefore \quad C_n = \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} dx P(x) He_n(x)}$$

using (6.1), we get:

$$C_0 = \int_{-\infty}^{\infty} dx P(x) = 1 \qquad \text{(normalization of PDF)}$$

$$
\begin{aligned}
C_1 &= -\int_{-\infty}^{\infty} dx x P(x) = 0 && \text{(zero mean)} \\
C_2 &= \frac{1}{2!}\int_{-\infty}^{\infty} dx(x^2-1)P(x) = \frac{1}{2}(\langle x^2\rangle - 1) \\
&= 0 && \text{(unit variance)} \\
C_3 &= \frac{-1}{3!}\int_{-\infty}^{\infty} dx(x^3-3x)P(x) = \frac{-1}{6}\langle x^3\rangle \\
&= \frac{-1}{6}k_3 && \text{(where } k_3 \text{ is skewness)} \\
C_4 &= \frac{1}{4!}\int_{-\infty}^{\infty} dx(x^4-6x^2+3)P(x) \\
&= \frac{1}{24}(\langle x^4\rangle - 3) \equiv \frac{1}{24}k_4 && \text{(where } k_4 \text{ is kurtosis.)}
\end{aligned}
$$

Therefore, the expansion coefficients are the moments of $P(x)$. Collecting all terms, the Gram-Charlier expansion of zero mean, unit variance PDF is given by:

$$
P(x) = G(x)\left[1 + \frac{1}{6}k_3 He_3(x) + \frac{1}{24}k_4 He_4(x) + \dots\right]
$$

6.2.4 Edgeworth Expansion

Next we extend to the case $\sigma \neq 1$. A useful expansion parameter is

$$
S_n = \frac{k_n}{2\sigma^{2n-2}}
$$

Note that S_n may carry units if k_n does.

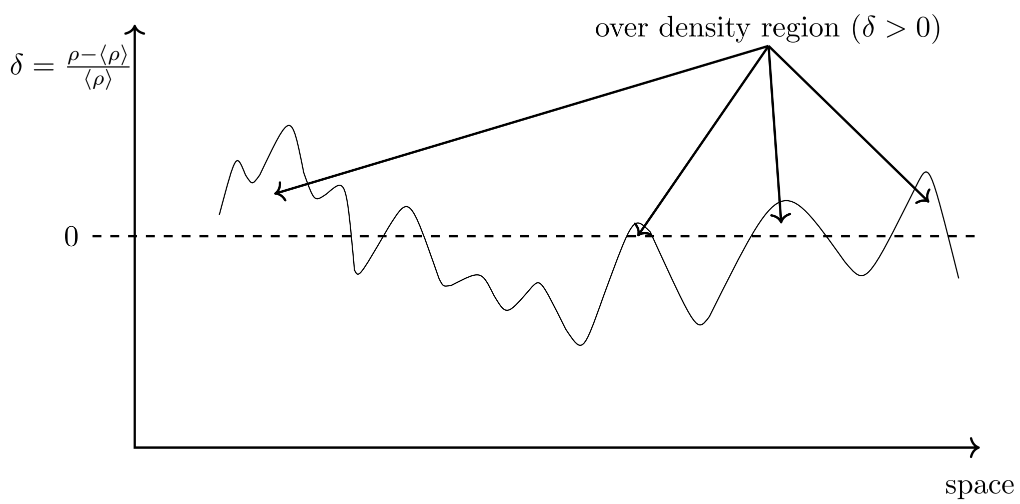
Let y be a random variable. The Edgeworth expansion becomes:

$$
\begin{aligned}
P(y) &= \frac{1}{\sigma}G(y/\sigma)\left[1 + \sigma\frac{S_3}{6}He_3(y\sigma) + \sigma^2\frac{S_4}{24}He_4(y\sigma) + \cdots\right] \\
&= \frac{1}{\sqrt{2\pi}\sigma}e^{-y^2/(2\sigma^2)}\left[1 + \sigma\frac{S_3}{6}\left(\frac{y^3}{\sigma^3} - 3\frac{y}{\sigma}\right) + \sigma^2\frac{S_4}{24}\left(\frac{y^4}{\sigma^4} - 6\frac{y^2}{\sigma^2} + 3\right) + \cdots\right]
\end{aligned}
$$

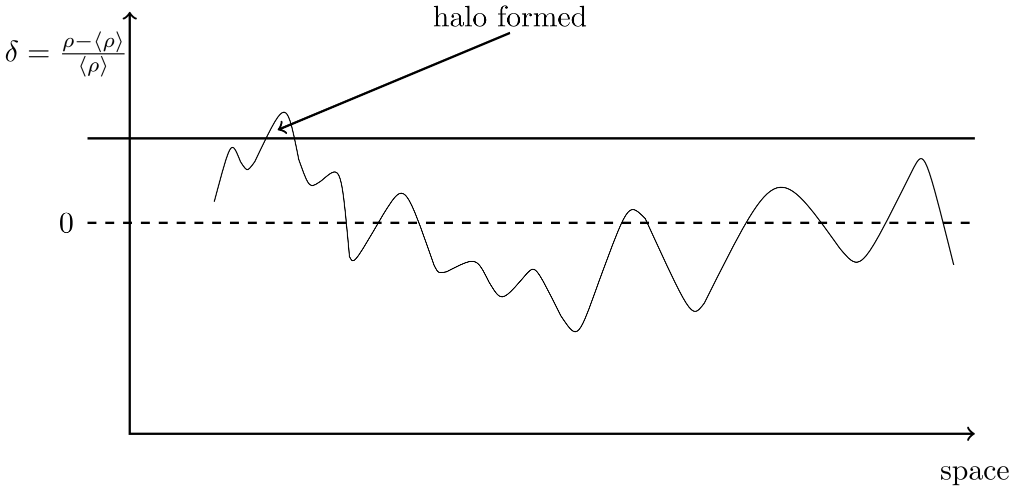
This is the Edgeworth Expansion.

6.3 Application : Gravitational Clustering

In cosmology, it is believed that collapsed structures like dark matter halos form at regions corresponding to high-density peaks. Therefore, within this framework, the statistical properties of such structures—like the number density of halos—can be inferred from the probability distribution function (PDF) of the underlying density field.



In the most basic model, halos are assumed to form in regions where the density contrast δ surpasses a critical threshold, typically taken to be $\delta_c = 1.686$.



What about mass? The standard method looks at early times when $\delta \ll 1$. The mass in a region of radius R is

$$M = \frac{4\pi}{3} \langle\rho\rangle R^3$$

We then smooth the density field with a top-hat filter of radius R—effectively binning it on that scale.

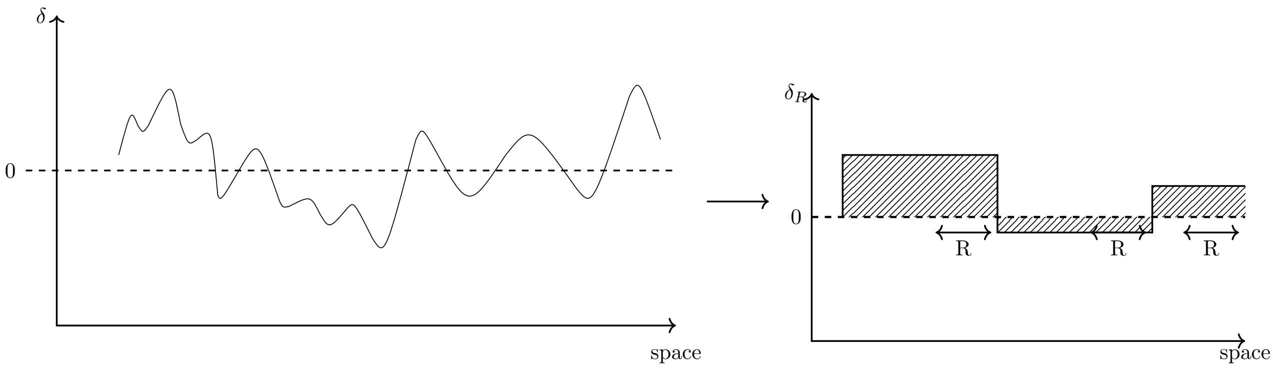
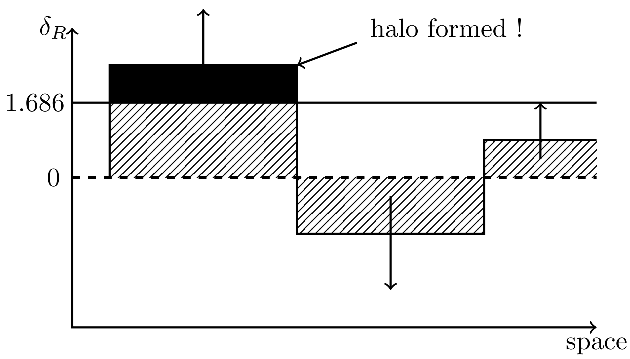


Figure 6.2: The density function is being smoothed out by using the top hat function.

Then track δ_R as it grows until it crosses the threshold $\delta_c = 1.686$. At that point, a collapsed object forms with mass

$$M = \frac{4\pi}{3} \langle\rho\rangle R^3.$$

According to Press & Schechter, the mass function (number density of halos per unit mass) is:



$$\frac{\mathrm{d}n}{\mathrm{d}M} = -2\frac{\langle\rho\rangle}{M}\frac{\mathrm{d}}{\mathrm{d}M}\int_{\delta_c}^{\infty} d\delta_R\, P(\delta_R)$$

For a Gaussian PDF, $P(\delta_R) = \frac{1}{\sqrt{2\pi}\sigma_R}\exp\left(-\frac{\delta_R^2}{2\sigma_R^2}\right)$, this gives:

$$\frac{\mathrm{d}n}{\mathrm{d}M} = 2\frac{\langle\rho\rangle}{M}\delta_c\frac{\mathrm{d}\sigma_R^{-1}}{\mathrm{d}M}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\delta_c^2}{2\sigma_R^2}\right)$$

This is the Press-Schechter mass function, normalized such that

$$\int_0^\infty dM\, M\frac{\mathrm{d}n}{\mathrm{d}M} = \langle\rho\rangle$$

i.e., total mass is accounted for by halos down to $M \to 0$, though this assumption is debatable.

Now consider an Edgeworth-expanded PDF:

$$
\begin{aligned}
P(\delta_R) = \frac{1}{\sqrt{2\pi}\sigma_R} \exp\left(-\frac{\delta_R^2}{2\sigma_R^2}\right) \Bigg[1 &+ \sigma_R \frac{S_3}{6}\left(\frac{\delta_R^3}{\sigma_R^3} - 3\frac{\delta_R}{\sigma_R}\right) \\
&+ \sigma_R^2 \frac{S_4}{24}\left(\frac{\delta_R^4}{\sigma_R^4} - 6\frac{\delta_R^2}{\sigma_R^2} + 3\right) + \cdots \Bigg]
\end{aligned}
$$

For non-Gaussianities, the mass function becomes:

$$
\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}M} = \sqrt{\frac{2}{\pi}} \frac{\langle\rho\rangle}{M} \exp\left(-\frac{\delta_c^2}{2\sigma_R^2}\right) \Bigg[&\delta_c \frac{\mathrm{d}\sigma_R^{-1}}{\mathrm{d}M}\left(1 + \sigma_R \frac{S_3}{6}\left\{\frac{\delta_c^3}{\sigma_R^3} - 2\frac{\delta_c}{\sigma_R} - \frac{\sigma_R}{\delta_c}\right\}\right) \\
&- \sigma_R \frac{1}{6}\frac{\mathrm{d}S_3}{\mathrm{d}M}\left(\frac{\delta_c^2}{\sigma_R^2} - 1\right) \Bigg]
\end{aligned}
$$

Positive skewness ($S_3 > 0$) enhances the abundance of massive halos.

6.4 Non-Gaussianity as a test of inflation

Having explored the mathematical framework for non-Gaussianity and its significance in cosmological perturbation theory, we now turn to its application in the Cosmic Microwave Background (CMB). We have already established that the primordial fluctuations imprinted on the CMB provide a unique window into the physics of the early universe. In this section, we focus to apply these ideas to the early universe and constrain the parameters of the cosmological collider physics. We have studied power spectrum and bispectrum in the earlier chapter, here we learn how they could be seen in data.

6.4.1 Power Spectrum

Inflation produces perturbations in the spatial part of the metric δg_{ij}. The scalar part of δg_{ij} will seed the structures we see today; while the tensor part of δg_{ij} will propagate as gravitational waves. Let us write[1]

$$
g_{ij} = e^{2\zeta}\gamma_{ij}a^2(t) \begin{cases} e^{2\zeta} &= 1 + 2\overset{\text{curvature perturbation}}{\zeta} + \dots \\ \gamma_{ij} &= \delta_{ij} + \underset{\text{gravitational wave}}{h_{ij}} + \dots \end{cases}
$$

where $i, j = 1, 2, 3$. In this section, we will study the statistical properties of ζ. It is convenient to recall the following relation between ζ and the observables.

1. CMB Temperature Anisotropy or very large angular scales where the "Sachs Wolfe" approximation is valid:
$$\frac{\delta T}{T} = -\frac{1}{5}\zeta(\hat{n}r_\star, z_\star) \qquad \text{where } r_\star \text{ is conformal distance to } z_\star$$

2. Newtonian gravitational potential in fourier space during matter era:
$$\tilde{\Psi}_{\vec{k}} = -\frac{3}{5}\tilde{\zeta}_{\vec{k}}\, T(k)$$
where $T(k)$ is the transfer function.

Inflation usually predicts a nearly power-law power spectrum.

$$\langle \tilde{\zeta}_{\vec{k}} \tilde{\zeta}^\star_{\vec{k}'} \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_\zeta(|\vec{k}|)$$

which can be written as:

$$P_\zeta(k) \equiv P_\zeta(|\vec{k}|) \propto k^{n_s - 4}$$

[1] we normally write

$$g_{ij} = e^{2\Phi}\gamma_{ij}a^2(t)$$

and then define ζ as

$$\zeta = \phi + \int \frac{d\rho}{3(\rho + p)}$$

But we use the uniform density gauge.

Inflation predicts that $n_s \approx 1$, and this is referred to as a "scale-invariant spectrum." To understand why $n_s \approx 1$ is called "scale-invariant," let's calculate the real-space 2-point correlation function:

$$\xi_\zeta(r) \equiv \langle \zeta(\vec{q})\zeta(\vec{q}+\vec{r})\rangle = \int \frac{d^3k}{(2\pi)^3} P_\zeta(k) e^{i\vec{k}\cdot\vec{r}}$$
$$= \int_0^\infty \frac{dk}{k} \frac{k^3 P_\zeta(k)}{(2\pi)^2} \frac{\sin(kr)}{kr}$$

Thus, the correlation function $\xi_\zeta(r)$ is given by

$$\boxed{\xi_\zeta(r) \approx \left.\frac{k^3 P_\zeta(k)}{2\pi^2}\right|_{k\approx\frac{1}{r}} \propto r^{1-n_s}}.$$

So, for $n_s \approx 1$, $\xi_\zeta(r)$ becomes independent of r. (More precisely, $\xi_\zeta(r)$ depends on r only logarithmically, which is why it's called "scale-invariant.") The latest observations give:

$$n_s = 0.96 \pm 0.01\,(68\%\,\mathrm{CL}).$$

To calculate the theoretical predictions for the power spectrum, one can use CAMB or its web interface. While the power spectra are often computed analytically in Fourier space, it's more natural to express statistical fluctuations in terms of the spherical harmonic basis when comparing theoretical predictions to real-world observables. The spherical harmonics Y_{lm} form a natural basis for functions $f(\hat{n})$ defined on the surface of a sphere, which is why they are a preferred representation for observations of the sky. Using the expansion coefficients a_{lm}, any arbitrary function $f(\hat{n})$ on the sphere can be expanded as:

$$f(\hat{n}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_{lm}(\hat{n}), \quad a_{lm} = \int_{4\pi} f(\hat{n}) Y^*_{lm}(\hat{n})\, d\Omega.$$

The observed isotropy of the universe enables the calculation of the corresponding power spectrum. For functions with rotational invariance, the coefficients are related to the angular power spectrum C_l by:

$$C_l \delta_{ll'}\delta_{mm'} = \langle a_{lm} a^*_{l'm'}\rangle,$$

where δ_{lm} is the Kronecker delta. Since we can only observe one realization of this underlying stochastic process, the measured value is the averaged quantity:

$$C_l = \frac{1}{2l+1}\sum_{m=-l}^{l} |a_{lm}|^2.$$

6.4.2 Bispectrum

Simple models of inflation satisfying all the following conditions:

- single field
- canonical kinetic term
- slow-roll and
- vacuum initial state

Produce a tiny amount of non-gaussianity-too liny to be detected by any experiments. Consider the bispectrum of the curvature perturbation at the end of inflation. It's fourier transform is given by:

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle = \int d^3x_1 d^3x_2 d^3x_3 e^{-i(\sum_{i=1}^{3}\vec{k}_i\cdot\vec{x}_i)}\langle \zeta(x_1)\zeta(x_2)\zeta(x_3)\rangle$$

Assuming the statistical homogeneity and isotropy, the correlation function is invariant under arbitrary spatial translation, so we set $x'_i = x_i - x_3$ for $i = 1, 2$

$$= \int d^3x'_1 d^3x'_2 d^3x_3 e^{-i(\sum_{i=1}^{2}\vec{k}_i\cdot(\vec{x'}_i+\vec{x}_3)+\vec{k}_3\cdot x_3)}\langle \zeta(x'_1)\zeta(x'_2)\zeta(0)\rangle$$

$$= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\int d^3x'_1 d^3x'_2 e^{-i(\sum_{i=1}^2 \vec{k}_i\cdot(\vec{x'}_i)}\langle\zeta(x'_1)\zeta(x'_2)\zeta(0)\rangle$$
$$= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B_\zeta(k_1,k_2,k_3)$$

the above simple models tend to produce

$$\frac{B_\zeta(k_1,k_2,k_3)}{P_\zeta(k_1)P_\zeta(k_2)+P_\zeta(k_1)P_\zeta(k_3)+P_\zeta(k_2)P_\zeta(k_3)} = \mathcal{O}(10^{-2})$$

which happens to be unobservably small.

However, if any of the above four conditions were violated, inflation could produce much larger non-Gaussianity. Since $B_\zeta(k_1,k_2,k_3)$ has 3 variables to play with, we have many shapes to play with. Imposing scale invariance allows us to reduce the number of degree of freedom to 3:

$$\frac{k_2}{k_1}, \quad \frac{k_3}{k_1}, \quad \text{and angle between } k_2 \text{ and } k_3$$

Among various shapes considered in the literature, let us pick 3 which are associated with the conditions mentioned earlier:

1. **Squeezed Limit**: $|\vec{k}_3| << |\vec{k}_2| \approx |\vec{k}_1|$

   Detection of this shape would rule out all single field models.

2. **Equilateral Limit**: $|\vec{k}_3| \approx |\vec{k}_2| \approx |\vec{k}_1|$

   Detection of this shape indicates enhanced field interaction at the horizon exit. This can be achieved by e.g. non-canonical kinetic terms.

3. **Folded Limit**: $|\vec{k}_3| \approx |\vec{k}_2| \approx |\vec{k}_1|/2$

   Detection of this shape would suggest that the initial state of quantum fluctuation in scalar fields were **not** in the vacuum state.

4. **More complex behavior**: When slow roll is violated, it often induces a violation of scale-invariance, and thus more complex shapes are possible, such as oscillating bispectrum.

For the rest of this chapter, we will focus only on "squeezed limit". Bispectrum estimation from CMB can be done using CMBBEST.

6.5 Multi-field inflation and Bispectrum

Perhaps, the most important goal of measuring the squeezed limit is to rule out the single-field models of inflation. How do we then calculate the bispectrum of the curvature perturbations, ζ, from multi-field inflation? For this purpose, there is a very convinent and powerful method called the δN formalism. Here, N is the number of e-folds of expansion:

$$N \equiv \ln\frac{a(t_{\text{observation}})}{a(t_{\text{initial}})}$$

Now, looking at the metric (spatial components, ignoring gravitational waves),

$$g_{ij} = a^2(t)e^{2\zeta(\vec{q})}\delta_{ij} \quad ; \qquad \vec{q} \text{ is the comoving coordinates}$$

One can define an inhomogeneous scale factor,

$$\hat{a}(t,q) = a(t)e^{\zeta(\vec{q})}$$

Then, if we choose the initial time to be on the flat hypersurface, in which $g_{ij} = a^2(t)\delta_{ij}$, then

$$\hat{N} = \ln\frac{\hat{a}(t_{\text{observation}})}{a(t_{\text{initial}})} = \ln\frac{a(t_{\text{observation}})}{a(t_{\text{initial}})} + \zeta(\vec{q})$$

Thus

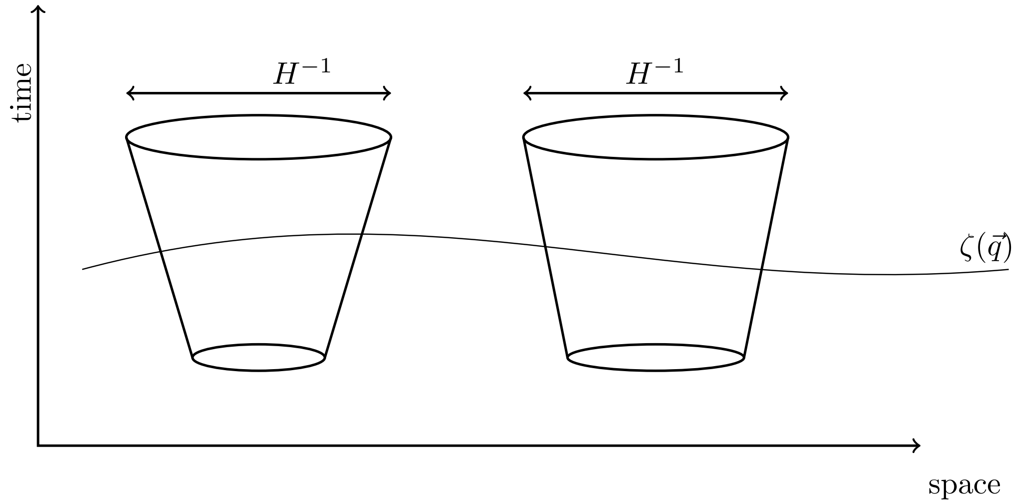
$$\zeta(\vec{q}) = \delta N(\vec{q}) \equiv \hat{N}(t_{\text{observation}}, t_{\text{initial}}, \vec{q}) - N(t_{\text{observation}}, t_{\text{initial}})$$

This is the δN formalism. To be more precise:

1. t_{initial} is on the flat hypersurface, in which $g_{ij} = a^2(t)\delta_{ij}$

2. $t_{\text{observation}}$ is on the uniform density hypersurface, in which $\delta\rho = 0$.

The next question is, "How do we compute $\delta N(\vec{q})$?" To calculate $\delta N(\vec{q})$, we can use "gradient expansion method", for which we systematically ignore the terms that involve spatial derivatives. This is a valid approximation, as long as the wavelength of perturbation we deal with is much greater than the Hubble length.



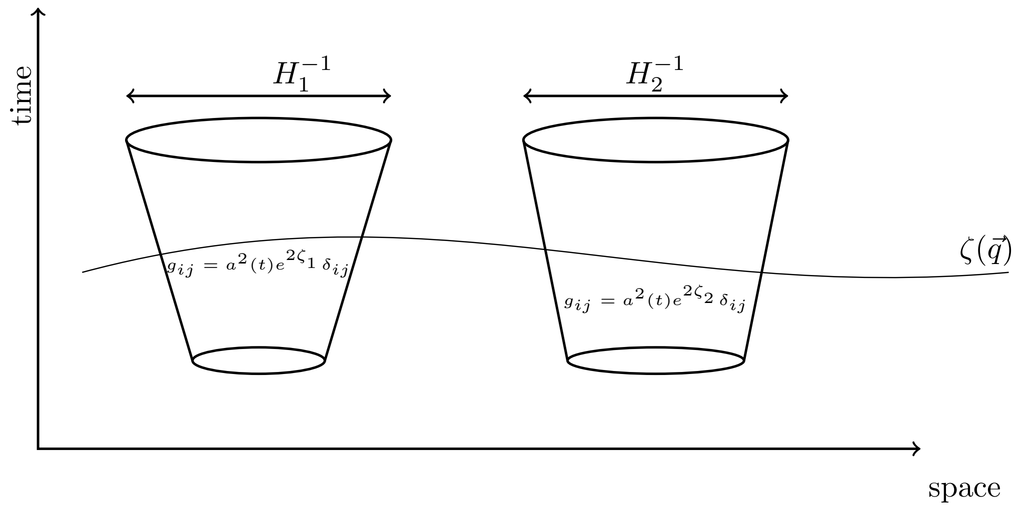
To zeroth order in gradient expansion, in which all the spatial derivatives are ignored, something remarkable happens. The Friedmann Equation for the backgroun (mean)

$$3M_{\text{pl}}^2 \langle H \rangle^2 = \langle \rho \rangle$$

is same as the Friedmann equation for the perturbed quantities

$$3M_{\text{pl}}^2 \hat{H}^2 = \hat{\rho}$$

This is a completely non trivial result. Naively, one can say "when the wavelength ofccurvature perturbation ζ is much larger than the Hubble horizon, each horizon patches evolves as if they were a separate universe.



Under this approximation, all we need to do is to solve the background evolution:

$$3M_{\text{pl}}^2 \langle H \rangle^2 = \langle \rho \rangle$$

for different initial conditions, and calculate perturbations as the differences between the initial conditions. Namely:

$$\begin{aligned}\hat{N}(\vec{q}) &= \hat{N}(\phi^I_{\text{initial}}(\vec{q}), \dot{\phi}^I_{\text{initial}}(\vec{q})) \\ &= \int dt \hat{H}(\phi^I_{\text{initial}}(\vec{q}), \dot{\phi}^I_{\text{initial}}, t)\end{aligned}$$

In order for ϕ to contribute to ρ (hence H), ϕ should be slow-rolling. Then, $\dot{\phi}$ is a function of ϕ, and thus we obtain:

$$\zeta(\vec{q}) = \delta N(\vec{q}) = \sum_I \frac{\partial N}{\partial \phi^I_{\text{initial}}} \delta\phi^I_{\text{initial}}(\vec{q}) + \frac{1}{2}\sum_{IJ} \frac{\partial^2 N}{\partial \phi^I_{\text{initial}} \partial \phi^J_{\text{initial}}} \delta\phi^I_{\text{initial}}(\vec{q}) \delta\phi^J_{\text{initial}}(\vec{q}) + \ldots$$

This completes the δN formalism.

6.6 Single Field Check

Let us apply the δN formalism to the single-field case, to make sure that we recover the well-known result:

$$\zeta = -\frac{H}{\dot{\phi}}\bigg|_{\text{initial}} \delta\phi_{\text{initial}}$$

Here, $\phi \equiv \phi^1$ because we have a single field model.

$$\hat{N} = \int_{t_i}^{t_{\text{obs}}} dt \hat{H} = \int_{\phi_{\text{ini}}}^{\phi_{\text{obs}}} d\phi \frac{H}{\dot{\phi}}$$

where we have introduced the short hand notation, $t_{\text{obs}} \equiv t_{\text{observation}}$ and $t_{\text{ini}} \equiv t_{\text{initial}}$. Now, t_{obs} is on the uniform density hypersurface, so

$$\phi_{\text{obs}} = \langle \phi \rangle_{\text{obs}}$$

On the other hand, t_{ini} is on the flat hypersurface, and thus

$$\phi_{\text{ini}} = \langle \phi \rangle_{\text{ini}} + \delta\phi_{\text{ini}}$$

Therefore,

$$\begin{aligned}
\hat{N} &= \int_{\langle \phi \rangle_{\text{ini}} + \delta\phi_{\text{ini}}}^{\langle \phi \rangle_{\text{obs}}} d\phi \frac{H}{\dot{\phi}} \\
&\approx \int_{\langle \phi \rangle_{\text{ini}}}^{\langle \phi \rangle_{\text{obs}}} d\phi \frac{\langle \dot{H} \rangle}{\dot{\phi}} \underbrace{- \frac{\langle H \rangle}{\dot{\phi}}\bigg|_{\text{ini}} \delta\phi_{\text{ini}} + \mathcal{O}(\delta\phi^2)}_{\delta N}
\end{aligned}$$

Hence,

$$\therefore \zeta = \delta N = \hat{N} - \langle N \rangle = -\frac{H}{\dot{\phi}}\bigg|_{\text{ini}} \delta\phi_{\text{ini}}$$

6.7 Local-form Non-Gaussianity

Look at the δN form:

$$\zeta(\vec{q}) = \delta N(\vec{q}) = \frac{\partial N}{\partial \phi^I} \delta\phi^I(\vec{q}) + \frac{1}{2} \frac{\partial^2 N}{\partial \phi^I \partial \phi^J} \delta\phi^I(\vec{q}) \delta\phi^J(\vec{q}) + \ldots$$

Here, we omitted the subscript "ini" for clarity and assumed that the repeated indices mean the summation. This form is "local" because the left hand side and the right hand side are evaluated at the same comoving position, $\vec{q}$. Also, this implies that ζ must be non-Gaussian even if $\delta\phi^I(\vec{q})$'s are Gaussian. To see this, compute the real space 3-point function:

$$\begin{aligned}
\langle \zeta(\vec{q}_1)\zeta(\vec{q}_2)\zeta(\vec{q}_3) \rangle = &\frac{\partial N}{\partial \phi^I} \frac{\partial N}{\partial \phi^J} \frac{\partial N}{\partial \phi^K} \left\langle \delta\phi^I(\vec{q}_1) \delta\phi^J(\vec{q}_2) \delta\phi^K(\vec{q}_3) \right\rangle \\
&+ \frac{1}{2} \frac{\partial N}{\partial \delta\phi^I} \frac{\partial N}{\partial \delta\phi^J} \frac{\partial^2 N}{\partial \delta\phi^K \partial \delta\phi^L} \left\langle \delta\phi^I(\vec{q}_1) \delta\phi^J(\vec{q}_2) \delta\phi^K(\vec{q}_3) \delta\phi^L(\vec{q}_3) \right\rangle \ldots
\end{aligned}$$

The 1^{st} term vanishes if $\delta\phi^I$'s are Gaussian. However, the 2^{nd} term does not (recall the wick's theorem). Now, let's compute the Bispectrum. In fourier space,

$$\zeta_{\vec{k}} = N_I \delta\phi^I_{\vec{k}} + \frac{1}{2} N_{IJ} \int \frac{d^3p}{(2\pi)^3} \delta\phi^I_{\vec{k}-\vec{p}} \phi^J_{\vec{p}} + \ldots$$

Here, we have dropped the ˜ as well as introduced a new notation, $N_I = \frac{\partial N}{\partial \phi^I}$, $N_{IJ} = \frac{\partial N}{\partial\delta\phi^I}\frac{\partial N}{\partial\delta\phi^I}\frac{\partial N}{\partial\delta\phi^J}$, etc.

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle = N_I N_J N_K \left\langle \delta\phi^I_{\vec{k}_1}\delta\phi^J_{\vec{k}_2}\delta\phi^K_{\vec{k}_3}\right\rangle + \frac{1}{2}N_I N_J N_{KL}\int \frac{d^3p}{(2\pi)^3}\left\langle \delta\phi^I_{\vec{k}_1}\delta\phi^J_{\vec{k}_2}\delta\phi^K_{\vec{k}_3-\vec{p}}\phi^L_{\vec{p}}\right\rangle$$
$$(2 - \text{permutations}) + \dots$$

Using wick's theorem, the second term becomes:

$$(2nd) = \frac{1}{2}N_I N_J N_K L\int \frac{d^3p}{(2\pi)^3}\Big\{\langle\delta\phi^I_{\vec{k}_1}\delta\phi^J_{\vec{k}_2}\rangle\langle\delta\phi^K_{\vec{k}_3-\vec{p}}\delta\phi^L_{\vec{p}_2}\rangle + \langle\delta\phi^I_{\vec{k}_1}\delta\phi^K_{\vec{k}_3-\vec{p}}\rangle\langle\delta\phi^J_{\vec{k}_2}\delta\phi^L_{\vec{p}}\rangle$$
$$+\langle\delta\phi^I_{\vec{k}_1}\delta\phi^L_{\vec{p}}\rangle\langle\delta\phi^J_{\vec{k}_2}\delta\phi^K_{\vec{k}_3-\vec{p}}\rangle\Big\}$$

Without loss of generality, we can write

$$\langle\delta\phi^I_{\vec{k}_1}\delta\phi^J_{\vec{k}_2}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2)\delta^{IJ}P_\phi(k_1)$$

Then,

$$(2nd) = \frac{1}{2}N_I N^I N^K_K(2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2)\delta^{(3)}(\vec{k}_3)P_\phi(k_1)P_\phi(k_3)\int d^3p\,1$$
$$+ N_I N_J N^{IJ}(2\pi)^3\delta^{(3)}(\vec{k}1+\vec{k}_2+\vec{k}_3)P_\phi(k_1)P_\phi(k_2)$$

However, the first line (involving $\delta^{(3)}(\vec{k}_3)$) vanishes because $P_\phi(k_3=0)=0$. Therefore, if $\delta\phi^I_{\vec{k}}$'s are all gaussian, then the leading order result is:

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)N_I N_J N^{IJ}(P_\phi(k_1)P_\phi(k_2) + \text{permutation})$$

On the other hand, the power spectrum of ζ is related to that of $\delta\phi$ as

$$\langle \zeta_{\vec{k}_1}\zeta^\star_{\vec{k}_2}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{k}_2)P_\zeta(\vec{k}_1) = N_I N_J\left\langle\delta\phi^I_{\vec{k}_1}\delta\phi^{J\star}_{\vec{k}_2}\right\rangle$$
$$= (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{k}_2)N_I N^I P_\phi(\vec{k}_1)$$

Hence,

$$\therefore P_\zeta(\vec{k}) = N_I N^I P_\phi(\vec{k})$$

writing the bispectrum as:

$$\langle \zeta_{\vec{k}_1}\zeta^\star_{\vec{k}_2}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}-3)B_\zeta(k_1,k_2,k_3)$$

we find,

$$\frac{B_\zeta(k_1,k_2,k_3)}{P_\zeta(k_1)P_\zeta(k_2)+\text{perm.}} = \frac{N_I N_J N^{IJ}}{(N_K N^K)^2}$$

Let us compute this in the single field limit ($\phi \equiv \phi^1$):

$$\frac{B_\zeta(k_1,k_2,k_3)}{P_\zeta(k_1)P_\zeta(k_2)+\text{perm.}} = \frac{\frac{\partial^2 N}{\partial\phi^2_{\text{ini}}}}{\left(\frac{\partial N}{\partial\phi_{\text{ini}}}\right)^2}$$

Here,

$$\frac{\partial N}{\partial\phi_{\text{ini}}} = -\frac{H}{\dot\phi}\bigg|_{\text{ini}} = -\frac{1}{M_{\text{pl}}}\frac{1}{\sqrt{2\epsilon}} \qquad \text{where } \epsilon = \frac{1}{2M_{\text{pl}^2}}\frac{\dot\phi^2}{H^2}$$

and thus,

$$\frac{\partial^2 N}{\partial\phi^2_{\text{ini}}} = \frac{1}{2}\frac{\partial\ln\epsilon}{\partial\phi_{\text{ini}}}$$

Therefore, for a single field model in which $\delta\phi$ is Gaussian, we obtain

$$\frac{B_\zeta(k_1,k_2,k_3)}{P_\zeta(k_1)P_\zeta(k_2)+\text{perm.}} = \frac{1}{2}\frac{\frac{\partial\ln\epsilon}{\partial\phi_{\text{ini}}}}{\frac{\partial N}{\partial\phi_{\text{ini}}}} = 2\epsilon-\eta$$

Indeed, we find ${}^{B_\zeta}/{P_\zeta}^2 \sim \mathcal{O}(10^{-2})$, which is too small to be observed, and thus detection of this quantity would rule out single-field slow-roll inflation models.

6.7.1 Squeezed Bispectrum

Since, $P_\zeta(k)$ is nearly scale invariant, it scales as $P_\zeta \propto 1/k^3$. This means that $B_\zeta(k_1, k_2, k_3)$ for the above local-form non-Gaussianity, peaks when one of the k_i's is very small. If we order $k_1 \geq k_2 \geq k_3$, then this bispectrum peaks when $k_3 << k_2 \approx k_1$.

For this limit, we find:

$$\begin{aligned} B_\zeta(k_1, k_2, k_3 \to 0) &\to \frac{2N_I N_J N^{IJ}}{(N_K N^K)^2} P_\zeta(k_1) P_\zeta(k_3) \\ &= 2(2\epsilon - \eta) P_\zeta(k_1) P_\zeta(k_3) \qquad \text{(for single field)} \end{aligned}$$

6.7.2 Single-Field Theorem (Consistency Condition)

Actually, the Squeezed-limit bispectrum is much more powerful than just ruling out single-field Slow-Roll model. It can be used to rule out ALL Single-field models,regardless of the models . "Single-field Theorem" states:

$$B_\zeta(k_1, k_2, k_3 \to 0) \to (1 - n_\sigma) P_\zeta(k_1) P_\zeta(k_3)$$

for All single-field inflation models! So, here we are gonna use slightly non-standard approach to show this relation. We are starting with this formula:

$$\begin{aligned} \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = N_I N_J N_K \langle \delta\phi^I_{\vec{k}_1} \delta\phi^J_{\vec{k}_2} \delta\phi^K_{\vec{k}_3} \rangle \\ + N_I N_J N^{IJ} (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) [P_\phi(K_1) + P_\phi(K_2) + (perm)] \end{aligned}$$

While, we've ignored the first term by assuming that $\delta\phi^I_s$ Gaussian. However, for single-field model, the second term is small (of order ϵ). But at the level of $\mathcal{O}(\epsilon)$, we cannot ignore the first term, as there exist interactions of $\delta\phi$ during inflation, making $\delta\phi$ slightly non-Gaussian.

While the precise computation of first term requires Quantum Field Theory (and was done self-consistently by Maldacena(2003)), we make a short cut by considering only the squeezed limit, $k_3 \approx k_2 \approx k_1$. To understand the situation better, let us call:

$$k_3 \to k_L (\text{for Long wavelength})$$

and

$$k_1, k_2 \to k_S (\text{for Short wavelength})$$

where L is much larger than H^{-1}, ζ is much smaller. Then, we can write first term as

$$\begin{aligned} \langle \delta\phi_{\vec{k}_1} \delta\phi_{\vec{k}_2} \delta\phi_{\vec{k}_3} \rangle &= \langle \delta\phi_{\vec{k}_S} \delta\phi_{\vec{k}_S} \delta\phi_{\vec{k}_L} \rangle \\ &= \langle \langle \delta\phi_{\vec{k}_S} \delta\phi_{\vec{k}_S} \rangle_L \delta\phi_{\vec{k}_L} \rangle \end{aligned}$$

Here, $\langle \delta\phi_{\vec{k}_S} \delta\phi_{\vec{k}_S} \rangle_L$ is the 2-point function of the Short mode, Given that there exist the Long Mode. In single field mode, there is only one dynamical degree of freedom ζ. Therefore, we need to compute $\langle \delta\phi_{\vec{k}_S} \delta\phi_{\vec{k}_S} \rangle$ in the presence of ζ_L. We know, $g_{ij} = a^2(t) e^{2\zeta_L}$, e^{ζ_L} amounts rescaling of co-moving coordinate. The result is

$$\boxed{\langle \delta\phi_{\vec{k}_S} \delta\phi_{\vec{k}_S} \rangle_L = \langle \delta\phi_{\vec{k}_S} \delta\phi_{\vec{k}_S} \rangle - \frac{\mathrm{d} \ln k_S^3 P_\phi(k_S)}{\mathrm{d} \ln k_S} P_\phi(k_S) \zeta_L}$$

For single-field result $P_\phi(k) \propto H^2/k^3$ and $dN = d\ln k$, we find

$$\frac{d \ln k^3 P_\phi(k)}{d \ln k} = \frac{d \ln H^2}{d\phi} \frac{d\phi}{dN} = -2\epsilon$$

With this result, we find

$$\begin{aligned} \left(\frac{\partial N}{\partial \phi_{\text{ini}}} \right)^3 \langle \delta\phi_{\vec{k}_1} \delta\phi_{\vec{k}_2} \delta\phi_{\vec{k}_3} \rangle &= \left(\frac{\partial N}{\partial \phi_{\text{ini}}} \right)^3 (+2\epsilon) P_\phi(k_1) \langle \zeta_{\vec{k}_1 + \vec{k}_2} \delta\phi_{\vec{k}_3} \rangle \\ &= +2\epsilon P_\zeta(k_1) P_\zeta(k_3) \times (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \qquad (for\ k_3 \to 0) \end{aligned}$$

Combining this with second term

$$\begin{aligned} \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle &\xrightarrow{k_3 \to 0} (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \times \{+2\epsilon P_\zeta(k_1) P_\zeta(k_3) + (4\epsilon - 2\eta) P_\zeta(k_1) P_\zeta(k_3)\} \\ &= (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \times (6\epsilon - 2\eta) P_\zeta(k_1) P_\zeta(k_3) \\ &= (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \times (1 - n_S) P_\zeta(k_1) P_\zeta(k_3) \end{aligned}$$

Therefore, if we find that the squeezed limit bispectrum has exceeded $1 - n_s = \mathcal{O}(10^{-2})$, all single field models are ruled out.

6.8 f_{NL}

The non-linear parameter, f_{NL}, is used to characterize $\frac{B_\zeta}{P_\zeta^2}$ as follows:

$$\frac{6}{5} f_{NL} = \frac{B_\zeta(k_1, k_2, k_3)}{P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_3)}$$

The current best bound on f_{NL} from the WMAP 7-year data is:

$$f_{NL} = 32 \pm 21 \quad (68\% \text{ CL})$$

or

$$-10 < f_{NL} < 74 \quad (95\% \text{ CL}).$$

Note: The factor of $\frac{6}{5}$ arises because f_{NL} was originally defined for curvature perturbations during the matter era in Newtonian Gauge, where $\phi = \frac{3}{5}\xi$. Thus, the definition is:

$$2 f_{NL} \equiv \frac{B_\zeta(k_1, k_2, k_3)}{P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_3)}.$$

6.9 Trispectrum and Multi-field consistency condition

If $f_{NL} >> 1$ is found by, e.g. Plank, what should we do? Single fields are gone, should we then study multi-field models? Is there any way we can rule out multi-field models also? Perhaps the 4-point function can be used to test multi-field models. Let us start from the local-form (δN) non-Gaussian.

$$\zeta = N_I \delta\phi^I + \frac{1}{2} N_{IJ} \delta\phi^I \delta\phi^J + \frac{1}{6} N_{IJK} \delta\phi^I \delta\phi^J \delta\phi^K + \dots$$

The 4-point function can be obtained from:

$$\begin{aligned} \tau_{NL} &: (1st) \times (1st) \times (2nd) \times (2nd) \\ g_{NL} &: (1st) \times (1st) \times (1st) \times (3rd) \end{aligned}$$

The local form non-Gaussianity then yields:

$$\begin{aligned} \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \\ \times \Big[\tau_{NL} P_\zeta(\vec{k}_1) P_\zeta(\vec{k}_2) \{ P_\zeta(|\vec{k}_1 + \vec{k}_3|) + P_\zeta(|\vec{k}_1 + \vec{k}_4|) \} + 10 \text{ perm} \\ + g_{NL} \frac{54}{25} [P_\zeta(\vec{k}_1) P_\zeta(\vec{k}_2) P_\zeta(\vec{k}_3) + 3 \text{ perm}] \Big] \end{aligned}$$

The τ_{NL} term peaks at "small diagonal limit", and the g_{NL} term peaks at "squeezed limit". Both τ_{NL} and g_{NL} can be given in terms of N_I, N_{IJ} etc. The leading order contributions[2] are:

$$\begin{aligned} \tau_{NL} &= \frac{(N_{IJ} N^J)(N^{IK} N_K)}{(N_L N^L)^3} \\ \frac{54}{25} g_{NL} &= \frac{N_{IJK} N^I N^J N^K}{(N_L N^L)^3} \end{aligned}$$

Where we have the factor $54/25$ in front of g_{NL} for Historical reasons, similar to $\frac{6}{5}$ for f_{NL}. Of these, τ_{NL} is of great interest. At this order, T_{NL} is given by N_I and N_{IJ}. On the other hand, f_{NL} is also given by a combination of N_I and N_{IJ}, and thus there should be a relation between the two. (Recall $\frac{6}{5} f_{NL} = \frac{N_{IJ} N^I N^J}{(N_L N^L)^{3/2}}$.) To see this, define the following new variables:

$$\begin{aligned} a_I &= \frac{N_{IJ} N^J}{(N_L N^L)^{3/2}} \\ b_I &= \frac{N_I}{(N_L N^L)^{1/2}} \end{aligned}$$

[2] If these terms vanish due to symmetry reasons, then one needs to go to next order terms

Then, we get

$$\frac{6}{5}f_{NL} = a_I b^I$$
$$\tau_{NL} = (a_I a^I)(b_J b^J)$$

The Cauchy-Schwarz inequality, $(a_I a^I)(b_J b^J) \geq (a_I b^I)^2$, gives

$$\tau_{NL} \geq \left(\frac{6}{5}f_{NL}\right)^2 \tag{6.3}$$

This is the (tree-level) Suyama-Tanaguchi inequality.[18]

Is $\tau_{NL} \geq \left(\frac{5}{6}f_{NL}\right)^2$ general?

The derivation of this relation relies on the "tree-level approximation," where:

$$\zeta = N_I \delta\phi^I + \frac{1}{2}N_{IJ}\delta\phi^I\delta\phi^J + \frac{1}{6}N_{IJK}\delta\phi^I\delta\phi^J\delta\phi^K + \dots$$

The bispectrum (f_{NL}) arises from $(1st)\times(1st)\times(2nd)$ terms, while τ_{NL} comes from $(1st)\times(1st)\times(2nd)\times(2nd)$.

However, for certain models (due to symmetry), these terms can vanish, and the leading order contributions may instead arise from:

$$f_{NL} : (2nd) \times (2nd) \times (2nd) \qquad\qquad T_{NL} : (2nd) \times (2nd) \times (2nd) \times (2nd)$$

In such cases, it is not immediately clear if any general inequalities exist between these terms. However, the latest studies suggest that $\tau_{NL} \geq \left(\frac{5}{6}f_{NL}\right)^2$ may still hold for general multi-field inflation theories. This inequality can serve as a key test for the inflationary mechanism generating fluctuations.

6.10 Measuring non-Gaussianity from CMB

To understand how we determine $f_{NL} = 32 \pm 21$ from the WMAP data, it is useful to first work through a simpler example: estimating the skewness, $k_3 = \langle x^3 \rangle$, for a zero-mean, unit-variance Gaussian distribution. Using the "Taylor-expanded" PDF (up to k_3), we have:

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-x^2}{2\sigma^2}}\left[1 + \frac{k_3}{6}\left(\frac{x^3}{\sigma^6} - 3\frac{x}{\sigma^4}\right) + \dots\right]$$

The best-fit k_3 can be found by maximizing this PDF with respect to k_3. This is essentially using Bayes' Theorem, interpreting $P(x|k_3)$ as $P(k_3|x)$:

$$\langle\frac{\partial \ln P(x)}{\partial k_3}\rangle = 0$$

Expanding $\ln P(x)$ up to k_3^2:

$$\ln P(x) = \text{const} + \frac{1}{6}\left(\frac{x^3}{\sigma^6} - 3\frac{x}{\sigma^4}\right)k_3 - \left[\frac{1}{6}\left(\frac{x^3}{\sigma^6} - 3\frac{x}{\sigma^4}\right)\right]^2\frac{k_3^2}{2} + \dots$$

Taking the derivative with respect to k_3 and setting it to zero, we get:

$$\langle\frac{1}{36}\left(\frac{x^6}{\sigma^{12}} - 6\frac{x^4}{\sigma^{10}} + 9\frac{x^2}{\sigma^8}\right)\rangle k_3 = \langle\frac{1}{6}\left(\frac{x^3}{\sigma^6} - 3\frac{x}{\sigma^4}\right)\rangle$$

On evaluating the left-hand side (using $\langle x^6 \rangle = 15\sigma^6$ and $\langle x^4 \rangle = 3\sigma^4$), we find:

$$\text{LHS} = \frac{1}{6}\frac{1}{\sigma^6}$$

Thus, the estimator for k_3 is:

$$\boxed{k_3 = \langle x^3 \rangle - 3\sigma^2\langle x \rangle}$$

For a zero-mean variable (which we assumed), the second term vanishes, and we obtain $k_3 = \langle x^3 \rangle$. In practice, we don't access the ensemble average directly, so the actual estimator becomes:

$$k_3 = \frac{\frac{1}{6}\sum_i\left(\frac{x_i^3}{\sigma_i^6} - 3\frac{x_i}{\sigma_i^4}\right)}{\frac{1}{6}\sum_i\frac{1}{\sigma_i^6}}$$

where i refers to the i-th measurement. This form is quite sensitive and directly applicable to measurements from the WMAP data.

PDF for CMB

In order to write down the PDF of CMB temperature (or polarization) anisotropy, we need to know how to generalize the Gram-Charlier (or Edgeworth) expansion to a multivariable case. First, let us start from a Gaussian PDF. Temperature anisotropy (i.e., Temperature in a given direction $\hat{n}$ minus the mean temperature), $\delta T(\hat{n})$, obeys

$$P(\{\delta T(\hat{n}_i)\}) = \frac{1}{(2\pi)^{N_{\rm pix}/2}|\xi|^{1/2}} e^{-\frac{1}{2}\sum_{ij}\delta T(\hat{n}_i)(\xi^{-1})_{ij}\delta T(\hat{n}_j)}$$

where,

$$\xi_{ij} = \langle \delta T(\hat{n}_i)\delta T(\hat{n}_j)\rangle$$

and

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

It is more convenient to work in "Fourier space", however, since we deal with a field on 2-sphere, we can not use the usual Fourier transformation on the full sky (unless we deal with a small section on the sky which may be approximated as flat). We thus use the spherical harmonic transform:

$$\delta T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

or

$$a_{lm} = \int d\hat{n}\delta T(\hat{n}) Y^\star_{lm}(\hat{n})$$

Then, a gaussian PDF becomes

$$P(\{a_{lm}\}) = \frac{1}{(2\pi)^{N_{\rm harm}/2}|C|^{1/2}} e^{-\frac{1}{2}\sum_{lm,l'm'} a^\star_{lm}(C^{-1})_{\substack{lm\\l'm'}} a_{l'm'}}$$

where

$$C_{\substack{lm\\l'm'}} = \langle a^\star_{lm} a_{l'm'}\rangle$$

For translationally and rotationally invariant temperature anisotropy, we have

$$C_{\substack{lm\\l'm'}} = C_l \delta_{ll'}\delta_{mm'}$$

However, even if the CMB itself is translationally and rotationally invariant, noise and foreground emission may not be. Thus, for generality we use $C_{lm,l'm'}$, without approximating it to be diagonal.

6.10.1 Multivariate Expansion of PDF

We can now "Taylor expand" a Gaussian PDF of a_{lm}.

$$\begin{aligned}
P(\{a_{\ell m}\}) = \frac{1}{(2\pi)^{N_{\rm harm}/2}|C|^{1/2}} \exp\left(-\frac{1}{2}\sum_{\substack{\ell m\\ \ell' m'}} a^*_{\ell m}(C^{-1})_{\substack{lm\\l'm'}} a_{l'm'}\right) \\
\times \left\{1 + \frac{1}{6}\sum_{\text{all } lm}\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}\rangle \left[(C^{-1}a)_{l_1m_1}(C^{-1}a)_{l_2m_2}(C^{-1}a)_{l_3m_3}\right]\right. \\
\left. -\frac{1}{6}\sum_{\text{all } lm}\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}\rangle \left[3(C^{-1})_{\substack{l_1m_1\\l_2m_2}}(C^{-1}a)_{l_3m_3}\right] + \dots\right\}
\end{aligned}$$

If we compare this with a uni-variate case:

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)\left[1 + \frac{\kappa_3}{6}\left(\frac{x^3}{\sigma^6} - 3\frac{x}{\sigma^4}\right) + \dots\right]$$

The correspondence is clear. The above PDF can be used to derive an estimator for the full bispectrum, $\langle a_{\ell_1m_1}a_{\ell_2m_2}a_{\ell_3m_3}\rangle$, or the "m-averaged bispectrum",

$$B_{\ell_1\ell_2\ell_3} \equiv \sum_{\text{all m}}\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}\langle a_{\ell_1m_1}a_{\ell_2m_2}a_{\ell_3m_3}\rangle$$

However, given a low signal-to-noise ratio, it is more useful to fit the data to a model bispectrum with a fixed shape (i.e., dependence of l_1, l_2, l_3), and find the amplitude, such as $f_{\rm NL}$. This is what we are going to do now. Let's say, we have a model shape $B_{l_1 l_2 l_3}$, such that

$$B_{l_1 l_2 l_3} = f_{NL} \mathcal{B}_{l_1 l_2 l_3}$$

Then, the estimator is given by

$$f_{NL} = \frac{\frac{1}{6}\sum_{\text{all } lm} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \mathcal{B}_{l_1 l_2 l_3} \left[(C^{-1}a)_{l_1 m_1} (C^{-1}a)_{l_2 m_2} (C^{-1}a)_{l_3 m_3} - 3 (C^{-1})_{\substack{l_1 m_1 \\ l_2 m_2}} (C^{-1}a)_{l_3 m_3} \right]}{\frac{1}{6}\sum_{\text{all } lm} \sum_{\text{all } l'm'} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \mathcal{B}_{l_1 l_2 l_3} (C^{-1})_{\substack{l_1 m_1 \\ l'_1 m'_1}} (C^{-1})_{\substack{l_2 m_2 \\ l'_2 m'_2}} (C^{-1})_{\substack{l_3 m_3 \\ l'_3 m'_3}} \mathcal{B}_{l'_1 l'_2 l'_3} \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}} \tag{6.4}$$

For a diagonal C_l,

$$f_{NL} = \frac{\frac{1}{6}\sum_{\text{all } lm} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \mathcal{B}_{l_1 l_2 l_3} \left[\frac{a_{l_1 m_1}}{C_{l_1}} \frac{a_{l_2 m_2}}{C_{l_2}} \frac{a_{l_3 m_3}}{C_{l_3}} - 3\delta_{l_1 l_2} \delta_{m_1 m_3} \frac{a_{l_3 m_3}}{C_{l_1} C_{l_2}} \right]}{\frac{1}{6}\sum_{\text{all } lm} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 \frac{\mathcal{B}^2_{l_1 l_2 l_3}}{C_{l_1} C_{l_2} C_{l_3}}}$$

Comparing this to the estimator for the skewness k_3

$$k_3 = \frac{\frac{1}{6}\sum_i \left(\frac{x_i^3}{\sigma_i^6} - 3\frac{x_i}{\sigma_i^4} \right)}{\frac{1}{6}\sum_i \frac{1}{\sigma_i^6}}$$

The correspondence is again clear. This is the estimator used by WMAP team to obtain $f_{NL} = 32 \pm 21$.

6.11 Relation between a_{lm} and $\zeta_{\vec{k}}$

We observe $\delta T(\hat{n})$ or (a_{lm}). However, we wish to obtain information about $\zeta_{\vec{k}}$ from the observed a_{lm}. How are they related?
Let us begin with the simplest case. On very large angular scales (larger than the sound horizon at $z_\star = 1090$), the Sach-Wolfe approximation may be used. It gives

$$\frac{\delta T(\hat{n})}{T} = -\frac{1}{5}\zeta(\hat{n} r_\star, z_\star)$$

where $r_\star$ is the comoving angular diameter distance to $z_\star$. From this, it is clear that are seeing only a slice of 3D field, $\zeta(\vec{q})$. For example, if we take a tiny section of the sky centered on a $\hat{z}$ direction:
We can perform the 2d Fourier transform:

$$\frac{\delta \tilde{T}}{T}(l) = \int d^2\theta \frac{\delta T}{T}(\vec{\theta}) e^{-i\vec{l}\cdot\vec{\theta}}$$

where $\vec{\theta} = (\theta\cos\phi, \theta\sin\phi)$. Then we find the relation between $\delta\tilde{T}(\vec{l})$ and $\tilde{\zeta}(\vec{k})$ as

$$\frac{\delta \tilde{T}}{T}(\vec{l}) = \int d^2\theta e^{-i\vec{l}\cdot\vec{\theta}} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\hat{n} r_\star} \left(-\frac{1}{5}\tilde{\zeta}_{\vec{k}} \right)$$

using $\hat{n} \approx (\vec{\theta}, 1)$,

$$= \int d^2\theta e^{-i\vec{l}\cdot\vec{\theta}} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}_\perp\cdot\vec{\theta} r_\star} e^{ik_\parallel r_\star} \left(-\frac{1}{5}\tilde{\zeta}_{\vec{k}} \right)$$

where $\vec{k} = (\vec{k}_\perp, k_\parallel)$. The integral over $\vec{\theta}$ yields a 2d Dirac delta function:

$$= \int \frac{d^3k}{(2\pi)^3} (2\pi)^3 \delta^2(\vec{k}_\perp r_\star - \vec{l}) e^{ik_\parallel r_\star} \left(-\frac{1}{5}\tilde{\zeta}_{\vec{k}} \right)$$

$$\therefore \quad \frac{\delta \tilde{T}}{T} = \frac{1}{r_\star^2} \int \frac{dk_\parallel}{2\pi} e^{ik_\parallel r_\star} \left[-\frac{1}{5}\tilde{\zeta}\left(\vec{k}_\perp = \frac{\vec{l}}{r_\star}, k_\parallel \right) \right]$$

From this result, it is clear that we are mostly sensitive to the $\vec{k}$ that are perpendicular to our line of sight, and informartion on $k_\parallel$ is smeared out by the integral. This is sad news: this limits the statistical power of CMB, which is really a 2-d object. It is possible to retrieve the full 3-d information by using the large scale structure density field, More later.

6.11.1 Full Sky

For the full sky analysis, we must work with spherical harmonics. Again in the Sach-Wolfe limit,

$$a_{lm} = \int d\hat{n} Y^{\star}_{lm}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\hat{n}r_\star} \left(-\frac{1}{5}\tilde{\zeta}_{\vec{k}}\right)$$

Now, using partial wave decomposition of plane waves (called "Rayleigh's Formula")

$$e^{i\vec{k}\cdot\hat{n}r_\star} = \sum_{lm} 4\pi(i)^l J_l(kr_\star) Y_{lm}(\hat{n}) Y^{\star}_{lm}(\hat{k})$$

we obtain,

$$a_{lm} = \int d\hat{n} Y^{\star}_{lm}(\hat{n}) \int \frac{d^3k}{(2\pi)^3} \left(-\frac{1}{5}\tilde{\zeta}_{\vec{k}}\right) \sum_{l'm'} 4\pi(i)^{l'} J_{l'}(kr_\star) Y_{l'm'}(\hat{n}) Y^{\star}_{l'm'}(\hat{k})$$

Using $\int d\hat{n} Y_{l'm'}(\hat{n}) Y^{\star}_{l'm'}(\hat{n}) = \delta_{ll'}\delta_{mm'}$ and summing over all l' and m',

$$\therefore \qquad a_{lm} = 4\pi i^l \int \frac{d^3k}{(2\pi)^3} J_l(kr_\star) Y^{\star}_{lm}(\vec{k}) \left(-\frac{1}{5}\tilde{\zeta}_{\vec{k}}\right)$$

The fact that we pick up $\vec{k}_\perp$ is obscured in this expression. Now let us go beyond the Sach-Wolfe limit, and include all the relevant effect. We replace

$$-\frac{1}{5} J_l(kr_\star) \to g^{(s)}_{Tl}(k)$$

and write

$$a_{lm} = 4\pi i^l \int \frac{d^3k}{(2\pi)^3} g^{(s)}_{Tl}(k)) Y^{\star}_{lm}(\vec{k}) \tilde{\zeta}_{\vec{k}}$$

where $g^{(s)}_{Tl}(k)$ is called "radiation transfer function", which can be computed using linear Boltzmann code such as CMBFast, CAMB etc. The power spectrum is given by:

$$C_l = \langle |a_{lm}|^2 \rangle = \frac{2}{\pi} \int k^2 dk P_\zeta(k) g^{(s)}_{Tl}(k)$$

6.11.2 Local-form CMB Bispectrum

In order to make a better contact with the literature, let us now relate a_{lm} to Φ (Curvature perturbation in matter era, Newtonian gauge):

$$a_{lm} = 4\pi i^l \int \frac{d^3k}{(2\pi)^3} \Phi_l(k) g^{(s)}_{Tl}(k) Y^{*}_{lm}(\hat{k})$$

Now, for the local-form bispectrum:

$$\left\langle \Phi_{\vec{k}_1} \Phi_{\vec{k}_2} \Phi_{\vec{k}_3} \right\rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \times 2f_{\rm NL} \left[P_\Phi(k_1) P_\Phi(k_2) + (\text{perm.})\right]$$

The CMB bispectrum is given by:

$$\langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3} \rangle = g^{l_1l_2l_3}_{m_1m_2m_3} \times 2f_{\rm NL} \int r^2 dr \left[\beta_{l_1}(r)\beta_{l_2}(r)\alpha_{l_3}(r) + (\text{perm.})\right]$$

where

$$g^{l_1l_2l_3}_{m_1m_2m_3} \equiv \int d\hat{n} Y_{l_1m_1}(\hat{n}) Y_{l_2m_2}(\hat{n}) Y_{l_3m_3}(\hat{n})$$

$$\alpha_l(r) \equiv \frac{2}{\pi} \int k^2 dk g_{Tl}(k) J_l(kr)$$

$$\beta_l(r) \equiv \frac{2}{\pi} \int k^2 dk P_\Phi(k) g_{Tl}(k) J_l(kr)$$

Using this in the estimator given in (6.4), we can estimate $f_{\rm NL}$ from the data!! In simpler models f_{NL} is scale independent and constant, however, as often in cosmological collider physics, f_{NL} will not just be scale dependent but also inherit certain oscillatory features from the bispectrum. The constant value just suggests that any truly physical imprint of that long mode on short scales would be suppressed by $(k_I/k_1)^2$ and is therefore extremely small. But in our study of three point and four point correlators, we had introduced a heavy spectator field that interacts with inflaton. In the squeezed limit, exchange of that extra field leaves a real, physical modulation of the short modes by the long mode. Mathematically you find a new term in the squeezed bispectrum that scales like $(k_I/k_1)^{3/2}$ not $(k_I/k_1)^2$. That's a larger effect in the limit $k_I \to 0$. With the 12cm tomography the prospects seem to be better, at least in theory, to probe new physics from the CMB.

Chapter 7

Discussion and Conclusion

Over the course of this thesis, we have built a coherent framework for understanding how quantum fields in an inflating spacetime give rise to the seeds of cosmic structure—and how those seeds retain tell-tale signatures of heavy particles and nontrivial interactions. Beginning in Chapter 2 with the in–in formalism, we developed the machinery to compute real-time expectation values in a time-dependent background. In Chapter 3 we revisited quantum field theory in curved spacetime, emphasizing Bogoliubov transformations and the observer-dependence of the vacuum. Chapter 4 then applied these ideas to gravitational particle production—first via analogies with a forced harmonic oscillator, then in explicit models (Bernard–Duncan and de Sitter space).

In Chapters 5 and 6, we showed how those very particles manifest in the statistics of the Cosmic Microwave Background. In particular:

- **Power spectrum** (two-point function): we recovered the nearly scale-invariant form characteristic of slow-roll inflation, with small corrections due to particle production and time-dependent masses.
- **Bispectrum** (three-point function): we derived explicit templates showing oscillatory "resonant" features whose frequency and phase encode the mass and spin of intermediate fields. These features deviate from the vanilla single-field consistency condition and open a direct window onto new high-energy degrees of freedom.
- **Trispectrum** (four-point function): we identified the collapsed-limit signal produced by pairwise exchange of heavy scalars, demonstrating how its momentum dependence sharpens and complements the bispectrum search.

In later chapters, we investigated how the particle production described earlier leaves an imprint in the higher-order correlation functions—particularly the bispectrum and trispectrum—showing that non-adiabatic evolution and Bogoliubov mixing generate characteristic oscillatory modulations and angular dependence. These "cosmological collider" signatures encode the mass spectrum and spin information of transient fields during inflation, turning the sky itself into a laboratory for high-energy physics.

Key Lessons and Implications

- **Observer-independence vs. physical imprint.** Although the notion of particles depends on the choice of vacuum (e.g. Minkowski vs. Rindler vs. de Sitter), the induced correlations in the curvature perturbation are physical and measurable in CMB anisotropies ($a_{\ell m} \leftrightarrow \zeta_{\mathbf{k}}$).
- **Non-Gaussianity as a collider.** Higher-point functions are not mere corrections to the Gaussian power spectrum, but carry orthogonal information: oscillatory shapes from heavy-field exchange, angular patterns from spin, and scale-dependence from nontrivial time evolution.
- **Complementarity of bispectrum and trispectrum.** While the bispectrum is often the first nonzero probe of interactions, the trispectrum in its collapsed limit can strongly enhance sensitivity to weakly coupled or very massive states.

Outlook

Current CMB experiments (e.g. *Planck*, ACTPol) have already placed interesting bounds on simple resonant bispectrum templates. However, next-generation surveys—Simons Observatory, CMB-S4, and LiteBIRD—will improve sensitivity to non-Gaussian oscillations by up to an order of magnitude. Moreover, large-scale structure surveys (e.g. DESI, Euclid) can access complementary squeezed-limit signals in galaxy clustering and weak lensing.

On the theoretical front, several avenues remain rich for exploration:

- *Loop corrections & stochastic effects:* understanding the robustness of resonant features under radiative corrections and infrared backreaction.
- *Beyond single-exchange diagrams:* extending templates to multiple interacting sectors, non-Abelian symmetries, or gauge field production.
- *UV completion:* embedding these phenomenological signals into string-inspired or quantum gravity frameworks, thereby tying observational templates to microscopic parameters.

In closing, this thesis has shown that by treating the early universe as a quantum detector, we can transform primordial correlation functions into precise probes of inflationary particle physics. As observational and theoretical tools continue to advance, the prospect of discovering heavy relics from the dawn of time grows ever more promising—bringing us closer to answering the age-old question of how the cosmos began and what fundamental laws governed its birth.

Appendix: Massive Field in de Sitter spacetime

The action for a massive scalar field in an FRW background is

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}m^2\chi^2\right] \\ &= \frac{1}{2}\int d\eta d^3x\left[a^2(\eta)\left((\chi')^2 - (\nabla\chi)^2\right) - m^2a^4(\eta)\chi^2\right]. \end{aligned}$$

Introducing the canonically-normalized field $u = a\chi$, and substituting $a(\eta) = -(H\eta)^{-1}$, the action becomes

$$\begin{aligned} S &= \frac{1}{2}\int d\eta d^3x\left[(u')^2 - (\nabla u)^2 - \left(m^2a^2 - \frac{a''}{a}\right)u^2\right] \\ &= \frac{1}{2}\int d\eta d^3x\left[(u')^2 - (\nabla u)^2 - \left(\frac{m^2/H^2 - 2}{\eta^2}\right)u^2\right]. \end{aligned}$$

The equation of motion (in Fourier space) then is

$$u_k'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2}\right)u_k = 0.$$

Defining $x \equiv -k\eta$, this becomes

$$x^2\frac{d^2u_k}{dx^2} + \left(x^2 - \nu^2 + \frac{1}{4}\right)u_k = 0, \quad \text{where } \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}.$$

Writing $u_k(\eta) \equiv \sqrt{x}\,g(x)$, this takes the form of a Bessel equation

$$x^2\frac{d^2g}{dx^2} + x\frac{dg}{dx} + (x^2 - \nu^2)g(x) = 0,$$

which has the following solution in terms of Hankel functions:

$$g(x) = c_1H_\nu^{(1)}(x) + c_2H_\nu^{(2)}(x).$$

Using

$$\begin{aligned} H_\nu^{(1)}(x) &\xrightarrow{x\to\infty} \sqrt{\frac{2}{\pi x}}\,e^{i(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \\ H_\nu^{(2)}(x) &\xrightarrow{x\to\infty} \sqrt{\frac{2}{\pi x}}\,e^{-i(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \end{aligned}$$

the early-time limit of the canonically-normalized field becomes

$$\lim_{-k\eta\to\infty} u_k(\eta) = \sqrt{\frac{2}{\pi}}\left(c_1e^{-\frac{i}{4}\pi(1+2\nu)}e^{-ik\eta} + c_2e^{\frac{i}{4}\pi(1+2\nu)}e^{ik\eta}\right).$$

This matches the Bunch-Davies initial condition,

$$\lim_{-k\eta\to\infty} u_k(\eta) = \frac{1}{\sqrt{2k}}e^{ik\eta},$$

if $c_1 = 0$ and $c_2 = e^{-i\frac{1}{4}\pi(1+2\nu)}\sqrt{\pi/(4k)}$, which fixes the solution to be

$$u_k(\eta) = e^{-\frac{i}{4}\pi(1+2\nu)}\sqrt{\frac{\pi}{4}}\sqrt{-\eta}H_\nu^{(2)}(-k\eta).$$

The de Sitter mode function for the massive field, $f_k = u_k/a$, then is

$$f_k = H\frac{\sqrt{\pi}}{2}e^{-\frac{i}{4}\pi(1+2\nu)}(-\eta)^{3/2}H_\nu^{(2)}(-k\eta) \qquad \text{(where } \nu = \sqrt{\tfrac{9}{4} - \tfrac{m^2}{H^2}}\text{)}$$

At late time:

$$\begin{aligned} f_k &= H\frac{\sqrt{\pi}}{2}e^{-\frac{i}{4}\pi(1+2\nu)}(-\eta)^{3/2}\frac{i}{\pi}\left[\Gamma(\nu)\left(\frac{-k\eta}{2}\right)^{-\nu} + e^{i\pi\nu}\Gamma(-\nu)\left(\frac{-k\eta}{2}\right)^{\nu}\right] \\ &= iH\sqrt{\frac{2}{\pi k^3}}e^{-\frac{i\pi}{4}}\left[e^{-i\frac{\pi\nu}{2}}\Gamma(\nu)\left(\frac{-k\eta}{2}\right)^{\frac{3}{2}-\nu} + e^{i\frac{\pi\nu}{2}}\Gamma(-\nu)\left(\frac{-k\eta}{2}\right)^{\frac{3}{2}+\nu}\right] \end{aligned}$$

For very massive fields $m >> H$, we have $\nu \approx {}^{im}/_{H}$

$$f_k = iH\sqrt{\frac{2}{\pi k^3}}e^{-\frac{i\pi}{4}}\left[e^{\pi m/2H}\Gamma\left(\frac{im}{H}\right)\left(\frac{-k\eta}{2}\right)^{\frac{3}{2}-\frac{im}{H}} + c.c\right]$$

In coordinate time:

$$\begin{aligned} f_k &= iH\sqrt{\frac{2}{\pi k^3}}e^{-\frac{i\pi}{4}}\left[e^{\pi m/2H}\Gamma\left(\frac{im}{H}\right)\left(\frac{ke^{-Ht}}{2H}\right)^{\frac{3}{2}-\frac{im}{H}} + c.c\right] \\ &= iH\sqrt{\frac{2}{\pi k^3}}e^{-\frac{i\pi}{4}}\left[e^{\pi m/2H}\Gamma\left(\frac{im}{H}\right)\left(\frac{k}{2H}\right)^{\frac{3}{2}-\frac{im}{H}} e^{-3Ht/2-imt} + c.c\right] \end{aligned}$$

Appendix: Conformal symmetry

In this appendix, we will discuss the concepts related to conformal bootstrap, where we essentially solve the conformal Ward identity, it is convenient to introduce conformal symmetry and the constraints it imposes on the correlation function beforehand. Then, we can discuss the derivation of conformal Ward identities.

Conformal Symmetry

Conformal symmetry is defined as an angle-preserving symmetry in the complex plane. However, this definition is not wrong but isn't very useful either. It requires us to define what constitutes an angle to fully comprehend it. There has been discussion in the literature where physicists have argued that such a definition only makes sense when we are only concerned with spatial directions. In special relativity and beyond, one must incorporate time as well, and the notion of an angle with time is not a very well-defined object. Therefore, an alternative that also holds for angle-preserving transformations is given by:

$$x \to \lambda x$$

Note that, this is **not** a general coordinate transformation which relabels the coordinates but it is actually changing the underlying geometry. Under this transformation, we observe:

$$\begin{aligned}\cos\theta &= \frac{\vec{x}\cdot\vec{y}}{|x||y|} \\ &= \frac{g_{\mu\nu}x^\mu y^\nu}{\sqrt{g_{\mu\nu}x^\mu x^\nu}\sqrt{g_{\mu\nu}y^\mu y^\nu}}\end{aligned}$$

under conformal transformation $g_{\mu\nu} \to \Omega(x) g_{\mu\nu}$

$$\begin{aligned}&= \frac{\Omega(x)g_{\mu\nu}x^\mu y^\nu}{\sqrt{\Omega(x)g_{\mu\nu}x^\mu x^\nu}\sqrt{\Omega(x)g_{\mu\nu}y^\mu y^\nu}} \\ &= \cos\theta\end{aligned}$$

Therefore, conformal transformation could be defined as one which scales the metric and as such preserves the angles.

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \qquad \text{(with } \Omega(x) > 0\text{)}$$

The approach to studying a Conformal Field Theory (CFT) is very different from that of Quantum Field Theory (QFT). In QFT, we typically write the Lagrangian first and then study the equations of motion. However, this is rarely the case in CFT. Instead, we focus little on writing the Lagrangian but rather utilize the conformal symmetry directly to guess the functional form of the correlation function. This approach is known as the conformal bootstrap.

Infinitesimal Conformal Transformation

The fundamental essence of conformal transformations resides in their infinitesimal form, which serves as a crucial tool for investigating how fields transform under these symmetries. It plays a pivotal role in defining the generator of the conformal group and, subsequently, constraining the set of possible correlators that are compatible with conformal symmetry. Any infinitesimal transformation can be expressed as:

$$x'^\mu = x^\mu + \underbrace{\epsilon^\mu(x)}_{\uparrow\,\text{infinitesimal}}$$

and subsequently,

$$x^\mu = x'^\mu - \epsilon^\mu(x)$$

therefore, the metric transforms like:

$$\begin{aligned}
g'_{\mu\nu} &= \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}}_{\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha(x)} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\
&= \left[\delta^\alpha_\mu - \frac{\partial \epsilon^\alpha(x)}{\partial x'^\mu}\right]\left[\delta^\beta_\nu - \frac{\partial \epsilon^\beta(x)}{\partial x'^\nu}\right] g_{\alpha\beta} \\
&= \delta^\alpha_\mu \delta^\beta_\nu g_{\alpha\beta} - \delta^\beta_\nu \partial_\mu \epsilon^\alpha(x) g_{\alpha\beta} - \delta^\alpha_\mu \partial_\nu \epsilon^\beta(x) g_{\alpha\beta} + \mathcal{O}(\epsilon^2) \\
\Omega(x) g_{\mu\nu} &= g_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu
\end{aligned}$$

In the third step, we used chain rule on $\epsilon^\alpha(x)$ and ignored $\mathcal{O}((\partial\epsilon)^2)$ terms. From the last line, it is reasonable to expect that:

$$\begin{aligned}
\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &\propto g_{\mu\nu} \\
\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) &= f(x) g_{\mu\nu}
\end{aligned} \tag{7.1}$$

Contracting Indicies

$$\begin{aligned}
\partial^\mu \epsilon_\mu(x) + \partial^\mu \epsilon_\mu(x) &= f(x) \delta^\mu_\mu \\
2(\partial \cdot \epsilon) &= \underset{\uparrow \text{dimension of spacetime}}{d}\, f(x) \\
f(x) &= \frac{2}{d} \frac{\partial \epsilon_\mu(x)}{\partial x_\mu}
\end{aligned}$$

Substituing back in (7.1)

$$\boxed{\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \frac{2}{d}(\partial \cdot \epsilon)\ g_{\mu\nu}} \tag{7.2}$$

Now, we operate by ∂^ν

$$\begin{aligned}
\frac{\partial}{\partial x'_\nu}\left[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)\right] &= \frac{\partial}{\partial x'_\nu}\left(\frac{2}{d}\partial \cdot \epsilon(x)\ \underset{\text{assuming flat metric}}{g_{\mu\nu}}\right) \\
\partial_\mu \underbrace{\partial^\nu \epsilon_\nu}_{\partial \cdot \epsilon} + \underbrace{\partial^\nu \partial_\nu}_{\Box} \epsilon_\mu &= \frac{2}{d} g_{\mu\nu} \partial^\nu \partial \cdot \epsilon \\
\partial_\mu(\partial \cdot \epsilon) + \Box \epsilon &= \frac{2}{d}\partial_\mu(\partial \cdot \epsilon)
\end{aligned}$$

Operating by ∂_ν

$$\begin{aligned}
\partial_\nu[\partial_\mu(\partial \cdot \epsilon) + \Box \epsilon] &= \partial_\nu\left[\frac{2}{d}\partial_\mu(\partial \cdot \epsilon)\right] \\
\left(1 - \frac{2}{d}\right)\partial_\nu \partial_\mu(\partial \cdot \epsilon) + \Box(\partial_\nu \epsilon_\mu) &= 0
\end{aligned} \tag{7.3}$$

under relabeling $\mu \leftrightarrow \nu$

$$\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box(\partial_\mu \epsilon_\nu) = 0 \tag{7.4}$$

adding (7.3) and (7.4)

$$\begin{aligned}
2\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box\underbrace{[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)]}_{\frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}} &= 0 \\
\left(1 - \frac{2}{d}\right)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \frac{1}{d}\Box(\partial \cdot \epsilon) g_{\mu\nu} &= 0 \\
\left[g_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu\right](\partial \cdot \epsilon) &= 0
\end{aligned} \tag{7.5}$$

Contracting the indicies

$$[d\Box + (d-2)\Box](\partial \cdot \epsilon) = 0$$

$$2(d-1)\Box(\partial\cdot\epsilon)=0$$

hence,

$$\boxed{(d-1)\Box(\partial\cdot\epsilon)=0} \tag{7.6}$$

if $d=1 \implies$ any $\epsilon^\mu(x)$ satisfies (7.6), therefore, is conformal transformation. It is interesting to note that any 1D QFT is conformal field theory, but for our purpose it's not very useful. We will be concerned with $d\neq 1$ for the rest of this notes unless stated otherwise. Consider the action of ∂_α on (7.2) and then cyclic relabeling of incidies as $\alpha\to\mu\to\nu$

$$\partial_\alpha[\partial_\mu\epsilon_\nu(x)+\partial_\nu\epsilon_\mu(x)]=\frac{2}{d}\partial_\alpha g_{\mu\nu}(\partial\cdot\epsilon) \tag{7.7}$$

$$\partial_\mu[\partial_\nu\epsilon_\alpha(x)+\partial_\alpha\epsilon_\nu(x)]=\frac{2}{d}\partial_\mu g_{\nu\alpha}(\partial\cdot\epsilon) \tag{7.8}$$

$$\partial_\nu[\partial_\alpha\epsilon_\mu(x)+\partial_\mu\epsilon_\alpha(x)]=\frac{2}{d}\partial_\nu g_{\alpha\mu}(\partial\cdot\epsilon) \tag{7.9}$$

Adding the first two equation and subtracting from the last, we get [(7.7) + (7.8) − (7.9)]:

$$\cancel{2}\partial_\alpha\partial_\mu\epsilon_\nu=\frac{\cancel{2}}{d}[g_{\mu\nu}\partial_\alpha+g_{\nu\alpha}\partial_\mu-g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon)$$

$$\partial_\alpha\partial_\mu\epsilon_\nu=\frac{1}{d}[g_{\mu\nu}\partial_\alpha+g_{\nu\alpha}\partial_\mu-g_{\alpha\mu}\partial_\nu](\partial\cdot\epsilon) \tag{7.10}$$

Therefore, the most general conformal transformation is of the type:

$$x'^\mu=x^\mu+\underbrace{a^\mu+b^\mu{}_\nu x^\nu+c^\mu{}_{\nu\alpha}x^\nu x^\alpha}_{\epsilon^\mu}$$

Where, $a^\mu, b^\mu{}_\nu$ and $c^\mu{}_{\nu\alpha}$ are parameters relevant to their transformation. The goal here is simple:

- First find the relevant transformations
- Then based on the transformation rule, find the generators.

For $\epsilon^\mu=a^\mu$:

$$\begin{aligned}x'^\mu&=x^\mu+a^\mu\\&=x^\mu+\delta^\mu_\nu a^\nu\\&=x^\mu+(\partial_\nu x^\mu)a^\nu\\&=[1+ia^\nu(-i\partial_\nu)]x^\mu\end{aligned}$$

Thus, the generator of translation is $P_\mu-i\partial_\mu$[1]. For $\epsilon^\mu=b^\mu{}_\alpha x^\alpha$, we refer to (7.2)

$$\begin{aligned}\partial_\mu\epsilon_\nu(x)+\partial_\nu\epsilon_\mu(x)&=\frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu}\\ \partial_\mu(b_{\nu\alpha}x^\alpha)+\partial_\nu(b_{\mu\alpha}x^\alpha)&=\frac{2}{d}(\partial^\mu b_{\mu\alpha}x^\alpha)g_{\mu\nu}\\ b_{\nu\alpha}\delta^\alpha_\mu+b_{\mu\alpha}\delta^\alpha_\nu&=\frac{2}{d}(b_{\mu\alpha}g^{\alpha\mu})g_{\mu\nu}\\ b_{\nu\mu}+b_{\mu\nu}&=\frac{2}{d}b^\alpha{}_\alpha g_{\mu\nu}\\ \frac{b_{\nu\mu}+b_{\mu\nu}}{2}&=\frac{1}{d}b^\alpha{}_\alpha g_{\mu\nu}\end{aligned}$$

now,

$$\begin{aligned}b_{\mu\nu}&=\frac{b_{\mu\nu}-b_{\nu\mu}}{2}+\frac{b_{\mu\nu}+b_{\nu\mu}}{2}\\&=M_{\mu\nu}+\lambda g_{\mu\nu}\end{aligned}$$

If $b_{\mu\nu}=\lambda g_{\mu\nu}(M_{\mu\nu}=0)$

$$x'^\mu=x_\mu+b^\mu{}_\nu x^\nu$$

[1]if we use $[1-a^\nu(\partial_\nu)]x^\mu$ as the definition, then $P_\mu=i\partial_\mu$ would be the generator

$$
\begin{aligned}
&= x^\mu + \lambda g^{\mu\alpha} \underbrace{g_{\alpha\nu} x^\nu}_{x_\alpha} \\
&= x^\mu + \lambda x^\mu \\
&= x^\mu + \lambda x^\nu \delta^\mu_\nu \\
&= x^\mu + \lambda x^\nu (\partial_\nu x^\mu) \\
&= x^\mu + i\lambda x^\nu (-i\partial_\nu x^\mu) \\
&= [1 + i\lambda(-i x^\nu \partial_\nu)] x^\mu
\end{aligned}
$$

Thus, the generator of dilatation is $D = -ix^\mu \partial_\mu$. For $b_{\mu\nu} = M_{\mu\nu}(\lambda = 0)$.

$$
\begin{aligned}
x'^\mu &= x^\mu + M^\mu{}_\nu x^\nu \\
&= x^\mu + M^\alpha{}_\nu \delta^\mu_\alpha x^\nu \\
&= x^\mu + M^\alpha{}_\nu (\partial_\alpha x^\mu) x^\nu \\
&= x^\mu + M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu \\
&= x^\mu + \frac{M_{\alpha\nu} - M_{\nu\alpha}}{2} (\partial^\alpha x^\mu) x^\nu \\
&= x^\mu + \frac{1}{2} M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu - \boxed{\frac{1}{2} M_{\nu\alpha} (\partial^\alpha x^\mu) x^\nu} \quad \text{relabeling } \nu \leftrightarrow \alpha \\
&= x^\mu + \frac{1}{2} M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu - \frac{1}{2} M_{\alpha\nu} (\partial^\nu x^\mu) x^\alpha \\
&= x^\mu + \frac{1}{2} M_{\alpha\nu} (x^\nu \partial^\alpha - x^\alpha \partial^\nu) x^\mu \\
&= x^\mu + \frac{i}{2} M_{\alpha\nu} \left\{ -i(x^\nu \partial^\alpha - x^\alpha \partial^\nu) \right\} x^\mu \\
&= x^\mu + \frac{i}{2} M_{\alpha\nu} \{ \underbrace{i(x^\alpha \partial^\nu - x^\nu \partial^\alpha)}_{L^{\alpha\nu}} \} x^\mu
\end{aligned}
$$

Thus, the generator of rotation is $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Now, the last part $\epsilon^\mu = c^\mu{}_{\nu\alpha} x^\nu x^\alpha = c^\mu{}_{\alpha\nu} x^\nu x^\alpha$, we refer to (7.10):

$$
\begin{aligned}
\partial_\alpha \partial_\mu \epsilon_\nu &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] (\partial \cdot \epsilon) \\
\partial_\alpha \partial_\mu (c_{\nu\sigma\beta} x^\sigma x^\beta) &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] \partial^\mu (c_{\mu\sigma\beta} x^\sigma x^\beta) \\
c_{\nu\sigma\beta} \partial_\alpha (\delta^\sigma_\mu x^\beta + x^\sigma \delta^\beta_\mu) &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] c_{\mu\sigma\beta} (g^{\sigma\mu} x^\beta + x^\sigma g^{\beta\mu}) \\
c_{\nu\sigma\beta} (\delta^\sigma_\mu \delta^\beta_\alpha + \delta^\sigma_\alpha \delta^\beta_\mu) &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] (c^\sigma{}_{\sigma\beta} x^\beta + c^\beta{}_{\sigma\beta} x^\sigma) \\
2 c_{\nu\mu\alpha} &= \frac{2}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] c^\sigma{}_{\sigma\beta} x^\beta \\
c_{\nu\mu\alpha} &= \underbrace{\frac{1}{d} c^\sigma{}_{\sigma\beta}}_{b_\beta} [g_{\mu\nu} \delta^\beta_\alpha + g_{\nu\alpha} \delta^\beta_\mu - g_{\alpha\mu} \delta^\beta_\nu] \\
&= g_{\nu\mu} b_\alpha + g_{\nu\alpha} b_\mu - g_{\mu\alpha} b_\nu
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\epsilon_\mu &= c_{\mu\alpha\beta} x^{\alpha\beta} \\
&= (g_{\mu\alpha} b_\beta + g_{\mu\beta} b_\alpha - g_{\alpha\beta} b_\mu) x^\alpha x^\beta \\
&= x_\mu (b \cdot x) + x_\mu (b \cdot x) - b_\mu (x \cdot x) \\
&= 2 x_\mu (b \cdot x) - x^2 b_\mu
\end{aligned}
$$

Hence, the Special Conformal Transformation looks like:

$$
\begin{aligned}
x'^\mu &= x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu \\
&= x^\mu + 2(b \cdot x) x^\nu \delta^\mu_\nu - x^2 b^\nu \delta^\mu_\nu \\
&= x^\mu + 2(b \cdot x) x^\nu \partial_\nu x^\mu - x^2 b^\nu \partial_\nu x^\mu
\end{aligned}
$$

$$= [1 + 2(b \cdot x)x^\nu \partial_\nu - x^2 b^\nu \partial_\nu] x^\mu$$
$$= [1 + \{2b^\alpha x_\alpha x^\nu \partial_\nu - x^2 b^\alpha \partial_\alpha\}] x^\mu$$
$$= [1 + ib^\alpha \{\underbrace{-i(2x_\alpha x^\nu \partial_\nu - x^2 \partial_\alpha)}_{K_\alpha}\}] x^\mu$$

Hence, the generator for Spatial Conformal Transformations (SCT) takes the form $K^\mu = -i(2x^\mu x \cdot \partial - x^2 \partial^\mu)$. We will now list all the **infinitesimal** transformations and their generators we found in this section.

1. Translation

$$x'^\mu = x^\mu + a^\mu \qquad\qquad P_\mu = -i\partial_\mu$$

2. Rotation

$$x'^\mu = x^\mu + M^\mu{}_\nu x^\nu \qquad\qquad L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

3. Dilatation

$$x'^\mu = (1 + \lambda) x^\mu \qquad\qquad D = -ix^\mu \partial_\mu$$

4. Special Conformal Transformation

$$x'^\mu = x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu \qquad\qquad K^\mu = -i(2x^\mu x \cdot \partial - x^2 \partial^\mu)$$

In the above listed transfomations, the parameters $a^\mu, M^\mu{}_\nu, \lambda$ and b^μ are all infinitesimal.

Finite Conformal Transformation

In the previous section, we considered the infinitesimal conformal transformation, however in this section we will consider the finite conformal transformation.

1. Translation

$$x'^\mu = x^\mu + \underset{\uparrow \text{finite vector}}{a^\mu} = e^{ia^\nu P_\nu} x^\mu$$

2. Dilatation

$$x'^\mu = \left(1 + \frac{\lambda}{N}\right) x^\mu$$

In order to achieve the finite dilatation, we use the infinitesimal transformation recursively by dividing the finite λ into infinitely many λ/N pieces and then transforming

$$x'^\mu = \left(1 + \frac{\lambda''}{N}\right) \underbrace{\left(1 + \frac{\lambda'}{N}\right)\left(1 + \frac{\lambda}{N}\right) x^\mu}_{x''^\mu}$$
$$= \lim_{N\to\infty} \left(1 + \frac{\lambda}{N}\right)^N x^\mu$$
$$= e^\lambda x^\mu = e^{i\lambda D} x^\mu$$

3. Rotation

$$x'^\mu = \left[1 + \frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta}\right]^\mu{}_\nu x^\nu$$
$$= \left[e^{\frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta}}\right]^\mu{}_\nu x^\nu = \Lambda^\mu{}_\nu x^\nu$$

4. The special conformal transformation

$$x'^\mu = x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu$$

infinitesimal parameter. ↓

let $b^\mu = t\, e^\mu$

$$x'^\mu(t) \equiv x^\mu(t) = x^\mu + 2t(e \cdot x) x^\mu - x^2 t e^\mu$$

To find the finite form of the transformation we have to recursively apply the above equation multiple times (Lie Algebra sence). The usual way is to integrate the infinitesimal form. The other way, and since we know that the transformations satisfy the conformal Killing equation, is to find the integral curve of the corresponding conformal Killing vector field as they are equivalent (Differential Geometry sense). Consider the $t-$derivative of the above[2].

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = 2(e\cdot x)x^\mu - x^2 e^\mu \tag{7.11}$$

defining $y^\mu(t) = \frac{x^\mu(t)}{x^2(t)}$

$$\begin{aligned}
\dot{y}^\mu(t) &\overset{\text{quotient rule}}{=} \frac{x^2\dot{x}^\mu - 2(\dot{x}\cdot x)x^\mu}{(x^2)^2} \\
&= \frac{x^2[2(e\cdot x)x^\mu - x^2e^\mu] - 2[2(e\cdot x)x^\nu - x^2e^\nu]x_\nu x^\mu}{x^4} \\
&= \frac{x^2[2(e\cdot x)x^\mu - x^2e^\mu] - 2[2(e\cdot x)x^2 - x^2(e\cdot x)]x^\mu}{x^4} \\
&= \frac{x^2[\cancel{2(e\cdot x)x^\mu} - x^2e^\mu] - \cancel{2(e\cdot x)x^2x^\mu}}{x^4} \\
\dot{y}^\mu(t) &= -e^\mu
\end{aligned}$$

Solving the above differential equation

$$\begin{aligned}
y^\mu(t) &= y^\mu(0) - te^\mu \\
\frac{x^\mu(t)}{x^2(t)} &= \frac{x^\mu(0)}{x^2(0)} - te^\mu
\end{aligned}$$

going back to the old notation $x'^\mu \equiv x^\mu(t)$

$$\begin{aligned}
\frac{x'^\mu}{x'^2} &= \frac{x^\mu}{x^2} - te^\mu \\
&= \frac{x^\mu}{x^2} - b^\mu
\end{aligned} \tag{7.12}$$

Squaring both sides

$$\begin{aligned}
\left(\frac{x'^\mu}{x'^2}\right)^2 \equiv \frac{x'^\mu}{x'^2}\frac{x'_\mu}{x'^2} &= \left(\frac{x^\mu}{x^2} - b^\mu\right)^2 \\
\frac{x'^2}{x'^4} &= \left(\frac{x^\mu}{x^2}\right)^2 + b^2 - \frac{2(x\cdot b)}{x^2} \\
\frac{1}{x'^2} &= \frac{1 + b^2x^2 - 2(x\cdot b)}{x^2} \\
x'^2 &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned} \tag{7.13}$$

referring to (7.12)

$$x'^\mu = x'^2\left[\frac{x^\mu}{x^2} - b^\mu\right]$$

and substituting (7.13)

$$\begin{aligned}
x'^\mu &= \frac{x^2}{1 - 2(x\cdot b) + b^2x^2}\left[\frac{x^\mu}{x^2} - b^\mu\right] \\
&= \frac{x^\mu - b^\mu x^2}{1 - 2(x\cdot b) + b^2x^2}
\end{aligned}$$

Above procedure also suggests that, finite SCT could be described as a sequence of inversion→translation→inversion. Where inversion is defined as:

$$I(x^\mu) = \frac{x^\mu}{x^2}$$

[2]when we consider the differential equation, we are no longer thinking of it as transformation but rather flow along a trajectory parameterized by t. This part was taken from pg 16 of [19]

Jacobian of the Transformation

$$\begin{aligned}
\text{Translation:} &\quad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1 \\
\text{Rotation:} &\quad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = 1 \\
\text{Dilataion:} &\quad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \lambda^{-D} \\
\text{Inversion:} &\quad \left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right| = \left(\frac{1}{\tilde{x}^2}\right)^D
\end{aligned}$$

How distances transform

Under translation

$$x'^\mu = x^\mu + a^\mu$$

So,

$$\begin{aligned}
x'^\mu_a - x'^\mu_b &= x^\mu_a + a^\mu - x^\mu_b - a^\mu \\
&= x^\mu_a - x^\mu_b
\end{aligned}$$

Thus, the distances are invariant under translation:

$$|x'_a - x'_b| = |x^\mu_a - x^\mu_b|$$

Under dilatation

$$x'^\mu = (1+\lambda)x^\mu$$

So,

$$\begin{aligned}
x'^\mu_a - x'^\mu_b &= (1+\lambda)x^\mu_a - (1+\lambda)x^\mu_b \\
&= (1+\lambda)(x^\mu_a - x^\mu_b)
\end{aligned}$$

We find that the distances between two point scales under dilatation, therefore the natural quantity which is invariant under both translation and dilatation is

$$\begin{aligned}
\frac{\left|x'^\mu_a - x'^\mu_b\right|}{\left|x'^\mu_c - x'^\mu_d\right|} &= \frac{\cancel{1+\lambda}\,|x^\mu_a - x^\mu_b|}{\cancel{1+\lambda}\,|x^\mu_c - x^\mu_d|} \\
&= \frac{|x^\mu_a - x^\mu_b|}{|x^\mu_c - x^\mu_d|}
\end{aligned}$$

Under special conformal transformation

$$\begin{aligned}
x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2(x \cdot b) + b^2 x^2} \\
&= \frac{x^\mu - b^\mu x^2}{\Lambda^2(x)}
\end{aligned}$$

So,

$$\begin{aligned}
x'^\mu_a &= \frac{x^\mu_a - b^\mu x^2_a}{\Lambda^2(x_a)} \\
x'^\mu_b &= \frac{x^\mu_b - b^\mu x^2_b}{\Lambda^2(x_b)}
\end{aligned}$$

and,

$$x'^\mu_a - x'^\mu_b = \frac{x^\mu_a - b^\mu x^2_a}{\Lambda^2(x_a)} - \frac{x^\mu_b - b^\mu x^2_b}{\Lambda^2(x_b)}$$

squaring both sides

$$
\begin{aligned}
(x_a'^\mu - x_b'^\mu)^2 &= \left(\frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}\right)^2 \\
&= \frac{x_a^2 + b^2(x_a^2)^2 - 2x_a^2(x_a \cdot b)}{\Lambda^4(x_a)} + \frac{x_b^2 + b^2(x_b^2)^2 - 2x_b^2(x_b \cdot b)}{\Lambda^4(x_b)} \\
&\quad - \frac{2}{\Lambda^2(x_a)\Lambda^2(x_b)}\left[x_a \cdot x_b - x_b^2(x_a \cdot b) - x_a^2(b \cdot x_b) + b^2 x_a^2 x_b^2\right] \\
&= x_a^2\left[\frac{1 - 2(x_a \cdot b)}{\Lambda^4(x_a)} + \frac{\overbrace{2(b \cdot x_b) - b^2 x_b^2}^{1-\Lambda^2(x_b)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1 - 2(x_b \cdot b)}{\Lambda^4(x_b)} + \frac{\overbrace{2(b \cdot x_a) - b^2 x_a^2}^{1-\Lambda^2(x_a)}}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2\left[\frac{1 - 2(x_a \cdot b) - \Lambda^2(x_a)}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{1 - 2(x_b \cdot b) - \Lambda^2(x_b)}{\Lambda^4(x_b)}\right. \\
&\quad \left. + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2\left[\frac{-b^2 x_a^2}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)}\right] + x_b^2\left[\frac{-b^2 x_b^2}{\Lambda^4(x_b)} + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)}\right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= \frac{(x_a - x_b)^2}{\Lambda^2(x_a)\Lambda^2(x_b)}
\end{aligned}
$$

Thus, we find that the ratio of distances are not invariant under SCT.

$$
\frac{\left|x_a'^\mu - x_b'^\mu\right|}{|x_a - x_b|} = \frac{1}{\Lambda(x_a)\Lambda(x_b)}
$$

where $\Lambda(x_a) = \sqrt{1 - 2(x_a \cdot b) + b^2 x_a^2}$ We can however, construct another quantity which is invariant under SCT.

$$
\begin{aligned}
\frac{|x_a' - x_b'|}{|x_b' - x_d'|}\frac{|x_d' - x_c'|}{|x_c' - x_a'|} &= \frac{\frac{|x_a - x_b|}{\Lambda(x_a)\Lambda(x_b)}}{\frac{|x_b - x_d|}{\Lambda(x_b)\Lambda(x_d)}}\frac{\frac{|x_d - x_c|}{\Lambda(x_d)\Lambda(x_c)}}{\frac{|x_c - x_a|}{\Lambda(x_c)\Lambda(x_a)}} \\
&= \frac{|x_a - x_b|}{|x_b - x_d|}\frac{|x_d - x_c|}{|x_c - x_a|}
\end{aligned}
$$

Such expressions are called, anharmonic ratios or cross-ratios.

Lie Algebra of Generators

$$
\begin{aligned}
[P_\mu, P_\nu] &= [-i\partial_\mu, -i\partial_\nu] \\
&= -[\partial_\mu, \partial_\nu] = 0
\end{aligned}
$$

Some useful identities

$$
\begin{aligned}
[x_\alpha, \partial_\beta]f &= x_\alpha \partial_\beta f - \underbrace{\partial_\beta(x_\alpha f)}_{(\partial_\beta x_\alpha)f + x_\alpha \partial_\beta f} \\
&= x_\alpha \partial_\beta f - x_\alpha \partial_\beta f - (\partial_\beta x_\alpha)f \\
&= -(\partial_\beta x_\alpha)f
\end{aligned}
$$

$$
\begin{aligned}
[x_\alpha, \partial_\beta] &= -\partial_\beta x_\alpha = -g_{\beta\alpha}\partial^\mu x_\alpha \\
&= g_{\beta\alpha}
\end{aligned} \tag{7.14}
$$

next is,

$$
\begin{aligned}
[x^2, \partial_\beta] &= [x^\alpha x_\alpha, \partial_\beta] \\
&= x^\alpha [x_\alpha, \partial_\beta] + [x^\alpha, \partial_\beta] x_\alpha \\
&= -x^\alpha g_{\beta\alpha} - \delta^\alpha_\beta x_\alpha \\
&= -x_\beta - x_\beta \\
&= -2x_\beta
\end{aligned} \tag{7.15}
$$

and the last one is,

$$
\begin{aligned}
[x_\mu x^\nu, \partial_\beta] &= x_\mu [x^\nu, \partial_\beta] + [x_\mu, \partial_\beta] x^\nu \\
&= -x_\nu \delta^\nu_\beta - x^\nu g_{\beta\alpha}
\end{aligned} \tag{7.16}
$$

We will now consider, the lie algebra of different operators one by one.

$$
\begin{aligned}
[P_\mu, D] &= [-i\partial_\mu, -ix^\alpha \partial_\alpha] \\
&= -[\partial_\mu, x^\alpha \partial_\alpha] \\
&= -x^\alpha [\partial_\mu, \partial_\alpha] - [\partial_\mu, x^\alpha] \partial_\alpha \\
&= -\delta^\alpha_\mu \partial_\alpha = -\partial_\mu = -i(-i\partial_\mu) \\
&= -iP_\mu
\end{aligned}
$$

$$
\begin{aligned}
[P_\mu, L_{\alpha\beta}] &= [-i\partial_\mu, -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)] \\
&= -[\partial_\mu, x_\alpha \partial_\beta - x_\beta \partial_\alpha] \\
&= -[\partial_\mu, x_\alpha] \partial_\beta + [\partial_\mu, x_\beta] \partial_\alpha \\
&= g_{\alpha\mu} \partial_\beta - g_{\beta\mu} \partial_\alpha \\
&= i(g_{\alpha\mu} P_\beta - g_{\beta\mu} P_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
[P_\mu, K_\nu] &= [-i\partial_\mu, -i(2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\nu)] \\
&= -[\partial_\mu, 2x_\nu x^\alpha \partial_\alpha - x^2 \partial_\nu] \\
&= -2x_\nu x^\alpha [\partial_\mu, \partial_\alpha] - 2[\partial_\mu, x_\nu x^\alpha] \partial_\alpha + x^2 [\partial_\mu, \partial_\nu] + [\partial_\mu, x^2] \partial_\nu \\
&= -2[\partial_\mu, x_\nu x^\alpha] \partial_\alpha + [\partial_\mu, x^2] \partial_\nu \\
&= -2(g_{\mu\nu} x^\alpha + \delta^\alpha_\mu x_\nu) \partial_\alpha + 2x_\mu \partial_\nu \\
&= -2g_{\mu\nu} x^\alpha \partial_\alpha - 2(x_\nu \partial_\mu - x_\mu \partial_\nu) \\
&= -2ig_{\mu\nu} D - 2iL_{\mu\nu} \\
&= -2i(g_{\mu\nu} D - L_{\mu\nu})
\end{aligned}
$$

$$
\begin{aligned}
[D, K_\mu] &= -[x^\alpha \partial_\alpha, 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu] \\
&= -2[x^\alpha \partial_\alpha, x_\mu x^\beta \partial_\beta] + [x^\alpha \partial_\alpha, x^2 \partial_\mu] \\
&= -2\left\{ x^\alpha [\partial_\alpha, x_\mu x^\beta] \partial_\beta + x_\mu x^\beta [x^\alpha, \partial_\beta] \partial_\alpha \right\} \\
&\quad + x^\alpha [\partial_\alpha, x^2] \partial_\mu + x^2 [x^\alpha, \partial_\mu] \partial_\alpha \\
&= -2\left\{ x^\alpha \left(g_{\alpha\mu} x^\beta + \delta^\beta_\alpha x_\mu \right) \partial_\beta + x_\mu x^\beta (-\delta^\alpha_\beta) \partial_\alpha \right\} \\
&\quad + 2x^2 \partial_\mu - \cancel{x^\alpha x^2 \partial_\alpha \partial_\mu} + \cancel{x^2 x^\alpha \partial_\alpha \partial_\mu} - x^2 \partial_\mu \\
&= -\cancel{2x_\mu x^\beta \partial_\beta} - 2x^\beta x_\mu \partial_\beta + \cancel{2x_\mu x^\beta \partial_\beta} + x^2 \partial_\mu \\
&= -(2x^\beta x_\mu \partial_\beta - x^2 \partial_\mu) \\
&= -iK_\mu
\end{aligned}
$$

$$
\begin{aligned}
[K_\mu, L_{\alpha\beta}] &= [-i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu), i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)] \\
&= [2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu, x_\alpha \partial_\beta - x_\beta \partial_\alpha] \\
&= 2[x_\mu x^\nu \partial_\nu, x_\alpha \partial_\beta] - [x^2 \partial_\mu, x_\alpha \partial_\beta] + \underbrace{2[x_\mu x^\nu \partial_\nu, x_\beta \partial_\alpha] - [x^2 \partial_\mu, x_\beta \partial_\alpha]}_{\alpha \leftrightarrow \beta}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\left\{x_\mu x^\nu[\partial_\nu, x_\alpha]\partial_\beta + x_\alpha[x_\mu x^\nu, \partial_\beta]\partial_\nu\right\} - x^2[\partial_\mu, x_\alpha]\partial_\beta - x_\alpha[x^2, \partial_\beta]\partial_\mu - (\alpha \leftrightarrow \beta) \\
&= \cancel{2x_\mu x^\nu (g_{\nu\alpha})\partial_\beta} - 2x_\alpha\left(g_{\mu\beta}x^\nu + \cancel{\delta^\nu_\beta x_\mu}\right)\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + 2x_\alpha x_\beta \partial_\mu - (\alpha \leftrightarrow \beta) \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + \cancel{2x_\alpha x_\beta\partial_\mu} + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2 g_{\mu\beta}\partial_\alpha - \cancel{2x_\beta x_\alpha\partial_\mu} \\
&= -2x_\alpha g_{\mu\beta}x^\nu\partial_\nu - x^2 g_{\mu\alpha}\partial_\beta + 2x_\beta g_{\mu\alpha}x^\nu\partial_\nu + x^2 g_{\mu\beta}\partial_\alpha \\
&= -g_{\mu\beta}(2x_\alpha x^\nu\partial_\nu - x^2\partial_\alpha) + g_{\mu\alpha}(2x_\beta x^\nu\partial_\nu - x^2\partial_\beta) \\
&= ig_{\mu\alpha}K_\beta - ig_{\mu\beta}K_\alpha = i(g_{\mu\alpha}K_\beta - g_{\mu\beta}K_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
[K_\mu, K_\nu] &= -[2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu, 2x_\nu x^\beta\partial_\beta - x^2\partial_\nu] \\
&= -4[x_\mu x^\alpha\partial_\alpha, x_\nu x^\beta\partial_\beta] + 2[x_\mu x^\alpha\partial_\alpha, x^2\partial_\nu] + 2[x^2\partial_\mu, x_\nu x^\beta\partial_\beta] - [x^2\partial_\mu, x^2\partial_\nu] \\
&= -4x_\nu x^\beta\left[x_\mu x^\alpha, \partial_\beta\right]\partial_\alpha - 4x_\mu x^\alpha\left[\partial_\alpha, x_\nu x^\beta\right]\partial_\beta + 2x_\mu x^\alpha\left[\partial_\alpha, x^2\right]\partial_\nu + 2x^2\left[x_\mu x^\alpha, \partial_\nu\right]\partial_\alpha \\
&\quad + 2x^2\left[\partial_\mu, x_\nu x^\beta\right]\partial_\beta + 2x_\nu x^\beta\left[x^2, \partial_\beta\right]\partial_\mu - x^2[\partial_\mu, x^2]\partial_\nu - x^2[x^2, \partial_\nu]\partial_\mu \\
&= \cancel{4x_\nu x^\beta(g_{\mu\beta}x^\alpha + \delta^\alpha_\beta x_\mu)\partial_\alpha} - \cancel{4x_\mu x^\alpha(g_{\alpha\nu}x^\beta + \delta^\beta_\alpha x_\nu)\partial_\beta} + 4x_\mu x^2\partial_\nu - 2x^2(\cancel{g_{\mu\nu}x^\alpha} + \delta^\alpha_\nu x_\mu)\partial_\alpha \\
&\quad + 2x^2(\cancel{g_{\mu\nu}x^\beta} + \delta^\beta_\mu x_\nu)\partial_\beta - 4x_\nu x^2\partial_\mu - 2x^2 x_\mu\partial_\nu + 2x^2 x_\nu\partial_\mu \\
&= \cancel{4x_\mu x^2\partial_\nu - 2x_\mu x^2\partial_\nu} + \cancel{2x_\nu x^2\partial_\mu - 4x_\nu x^2\partial_\mu} - \cancel{2x^2 x_\mu\partial_\nu} + \cancel{2x^2 x_\nu\partial_\mu} \\
&= 0
\end{aligned}
$$

Next, we will see that Conformal Algebra in d dimensions is isomorphic to the Lie algebra of the Lorentz group in $d+2$ dimensions, any conformal covariant correlator in d dimensions should be obtainable from Lorentz covariant expressions in $d+2$ dimensions via some kind of dimensional reduction procedure. This is essentially the idea behind **Embedding Formalism**. We define the following set of new operators:

$$
\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu} \\
J_{0,\mu} &= \frac{1}{2}\left(P_\mu + K_\mu\right) \\
J_{-1,\mu} &= \frac{1}{2}\left(P_\mu - K_\mu\right) \\
J_{-1,0} &= D
\end{aligned}
$$

with the property that

$$J_{ab} = -J_{ba}$$

where

$$a, b \in \{-1, 0, 1, \cdots, \overset{\text{d is dimension of spacetime}}{d}\}$$

These new generators, obey $SO(d+1,1)$ lie algebra:

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{bc}J_{ad}\right) \tag{7.17}$$

In this section, we will explicitly assume the form of flat metric as being euclidean, and given as:

$$g_{\mu\nu} = \eta_{\mu\nu} = (\underbrace{1, 1, \cdots, 1}_{d})$$

Our metric in (7.17) would be given as:

$$\eta_{ab} = (\underset{\eta_{-1-1}=-1}{-1}, \underset{\eta_{00}=1}{1}, \underbrace{1, \cdots, 1}_{\mu,\nu}) \tag{7.18}$$

If our original metric was Minkowski, we would have had:

$$\eta_{ab} = (-1, 1, \underbrace{-1, \cdots, 1}_{d})$$

We will now check, if (7.17) holds true:

$$[J_{\mu\nu}, J_{0,\alpha}] = \left[L_{\mu\nu}, \frac{1}{2}(P_\alpha + K_\alpha)\right]$$

$$
\begin{aligned}
&= \frac{1}{2}[L_{\mu\nu}, P_\alpha] + \frac{1}{2}[L_{\mu\nu}, K_\alpha] \\
&= -\frac{1}{2}[P_\alpha, L_{\mu\nu}] - \frac{1}{2}[K_\alpha, L_{\mu\nu}] \\
&= -\frac{1}{2}\left(\eta_{\alpha\mu}P_\mu - \eta_{\alpha\nu}P_\mu\right) - \frac{1}{2}(\eta_{\alpha\mu}K_\nu - \eta_{\alpha\nu}K_\nu) \\
&= -\eta_{\alpha\mu}\left[\frac{1}{2}(P_\nu + K_\nu)\right] + \eta_{\alpha,\nu}\left[\frac{1}{2}(P_\mu + K_\mu)\right] \\
&= -i\eta_{\alpha\mu}J_{0,\nu} + i\eta_{\alpha\nu}J_{0,\mu}
\end{aligned}
$$

$$
\begin{aligned}
[J_{0,\mu}, J_{-1,0}] &= \left[\frac{1}{2}(P_\mu + K_\mu), D\right] \\
&= \frac{1}{2}[P_\mu, D] + \frac{1}{2}[K_\mu, D] \\
&= -\frac{1}{2}iP_\mu - \frac{1}{2}(-iK_\mu) = \frac{-i}{2}(P_\mu - K_\mu) = -iJ_{-1,\mu}
\end{aligned}
$$

If we assume that the metric in (7.17) is indeed given by (7.18). Then, the algebra (7.17) holds true. This shows the isomorphism between the conformal group in $d-$dimensions and the group $SO(d+1,1)$ with $1/2(d+1)(d+2)$ parameters.

Conformal Generators on the Field

Finite form of conformal transformation $(x' = \Lambda x)$[3]

$$
\begin{aligned}
\Phi'_a(x') &= U(\Lambda)\Phi_a(x)U^{-1}(\Lambda) \\
\Phi'_a(\Lambda x) &= \sum_b \pi_{ab}(\Lambda)\Phi_b(x) \\
&= \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x') \\
&= \pi_{ab}(e^{i\omega_g c_g})\Phi_b(e^{-i\omega_g c_g}x')
\end{aligned}
\tag{7.19}
$$

We have dropped the $\sum$ sign and summation over repeated indices are implied. Infinitesimal form of (7.19):

$$
\begin{aligned}
\Phi'_a(x') &= (1 - i\omega_g \underbrace{T_g}_{\text{generator only acting on field}})_{ab}\Phi_b(\Lambda^{-1}x') \\
&= (1 - i\omega_g T_g)_{ab}\underbrace{\Phi_b[(1 - i\omega_g \overbrace{c_g}^{\text{generator which only acts on } x'^\mu})x'^\mu]}_{\Phi_b(x') + \{(1 - i\omega_g c_g)x'^\mu - x'^\mu\}\partial_\mu\Phi_b(x')} \\
&= (1 - i\omega_g T_g)_{ab}[\Phi_b(x') - i\omega_g c_g x'^\mu\partial_\mu\Phi_b(x')] \\
\Phi'(x') &= \Phi(x') - i\omega_g\left[T_g + \underbrace{c_g x'^\mu\frac{\partial}{\partial x'^\mu}}_{\text{accounts for the change in argument of field}}\right]\Phi(x') + \mathcal{O}(\omega_g^2)
\end{aligned}
$$

However, we will not use this approach but rather we will consider the transformations at origin and then translate it to every other point (method of induced representation). This approach is based on studying the stabilizer subgroup of the Conformal Symmetry.[4] So, if we study the same at origin:

$$
\begin{aligned}
\Phi'(0) &= \Phi(x') - i\omega_g\left[T_g + c_g x'^\mu\frac{\partial}{\partial x'^\mu}\right]\Phi(x')\Big|_{x'=0} \\
&= \Phi(0) - i\omega_g T_g\Phi(0)
\end{aligned}
$$

using translation operator

$$
\begin{aligned}
e^{ix^\lambda P_\lambda}\Phi'(0)e^{-ix^\alpha P_\alpha} &= e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} - e^{ix^\lambda P_\lambda}i\omega_g T_g\Phi(0)e^{-ix^\alpha P_\alpha} \\
\Phi'(x) &= \Phi(x) - e^{ix^\lambda P_\lambda}i\omega_g T_g e^{-ix^\sigma P_\sigma}e^{ix^\beta P_\beta}\Phi(0)e^{-ix^\alpha P_\alpha} \\
&= \Phi(x) - i\omega_g \underbrace{e^{ix^\lambda P_\lambda}T_g e^{-ix^\sigma P_\sigma}}_{\text{we will find these “translated operators” later}}\Phi(x)
\end{aligned}
$$

[3]tobias osborne's lecture notes pg 18

[4]pg 7 of "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications"

For translation

$$\Phi'(x+a) = e^{ia^\lambda P_\lambda}\Phi(x)e^{-ia^\alpha P_\alpha}$$
$$= e^{ia^\lambda [P_\lambda,\ ]}\Phi(x)$$

using (7.21)

$$= e^{a\cdot\partial}\Phi(x)$$

For rotation, at $x'^\mu = 0. \implies x^\mu = 0$

$$\Phi'_a(0) = \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}0) = \pi_{ab}(\Lambda)\Phi_b(0)$$

Now, assuming the generator of rotation $T_g = L_{\mu\nu}$ acts like[5]

$$L_{\mu\nu}\Phi_a(0) = S_{\mu\nu}\Phi_a(0) \tag{7.20}$$

at origin. At any other point, it will behave as:

$$\begin{aligned} L_{\mu\nu}\Phi_a(x) &= e^{ix^\beta P_\beta}L_{\mu\nu}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\ &= \underbrace{e^{ix^\beta P_\beta}L_{\mu\nu}e^{-ix^\sigma P_\sigma}}_{?}\underbrace{e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}}_{\Phi_a(x)} \end{aligned}$$

by taking the derivative of second term, we obtain the following commutator

$$\begin{aligned} \Phi_a(x) &= e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} \\ \partial_\mu\Phi_a(x) &= (\partial_\mu e^{ix^\lambda P_\lambda})\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)(\partial_\mu e^{-ix^\alpha P_\alpha}) \\ &= iP^\mu e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha} + e^{ix^\lambda P_\lambda}\Phi_a(0)e^{-ix^\alpha P_\alpha}(-iP^\mu) \\ &= iP^\mu\Phi_a(x) - i\Phi_a(x)P^\mu \\ &= i[P^\mu, \Phi_a(x)] \end{aligned} \tag{7.21}$$

We will now derive the form of $e^{ix^\beta P_\beta}L_{\mu\nu}e^{-ix^\sigma P_\sigma}$[6]:

$$\begin{aligned} e^{ix^\beta P_\beta}L_{\mu\nu}e^{-ix^\sigma P_\sigma} &= L_{\mu\nu} + [L_{\mu\nu}, -ix^\alpha P_\alpha] + \frac{1}{2!}[[L_{\mu\nu}, -ix^\alpha P_\alpha], -ix^\alpha P_\alpha] + \dots \\ &= L_{\mu\nu} + ix^\alpha\underbrace{[P_\alpha, L_{\mu\nu}]}_{i(g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu)} + \dots \\ &= L_{\mu\nu} + i^2x^\alpha(g_{\alpha\mu}P_\nu - g_{\alpha\nu}uP_\mu) \\ &= L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \\ &= L_{\mu\nu} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{we found in section 7}} \end{aligned}$$

we know, at $x' = 0$ we have $L_{\mu\nu} = S_{\mu\nu}$, so for the sake of consistency we get

$$e^{ix^\beta P_\beta}L_{\mu\nu}e^{-ix^\sigma P_\sigma} = \underbrace{S_{\mu\nu}}_{\text{Spin Operator}} + \underbrace{i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{\text{transforms the argument of field}}$$

The exponential map of above can be found in any textbook on QFT which describes rotation or Lorentz transformation.[7] If we ignore $S_{\mu\nu}$, then we can see how the last part acts on field:

$$\begin{aligned} x'^\mu &= \left(\delta^\mu_\nu + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right)x^\mu \\ &= x^\mu + \frac{i}{2}\omega_{\alpha\beta}i\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right)x^\nu \\ \Phi'(x) &= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\Phi(x) \end{aligned}$$

[5]pg 10, paragraph 2 of [20]
[6]using BCH lemma $e^ABe^{-A} = e^{[A,\ ]}B$
[7]check eqn 1.141 and 1.150 of [4]

$$
\begin{aligned}
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta} i(x^\alpha \partial^\beta - x^\beta \partial^\alpha)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha g^{\beta\sigma}\partial_\sigma - x^\beta g^{\alpha\sigma}\partial_\sigma)\Phi(x) \\
&= \Phi(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha \partial^\beta - x^\beta \partial^\alpha) x^\sigma \partial_\sigma \Phi(x) \\
&= \Phi(x) - \frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta} x \cdot \partial \Phi(x) \\
&\approx \Phi\left(x^\mu - \frac{i}{2}\omega_{\alpha\beta} L^{\alpha\beta} x^\nu\right) \\
\Phi'(x') &= \Phi(x)
\end{aligned}
$$

For dilatation, at $x'^\mu = 0$, $x'^\mu = (1+\lambda)x^\mu = 0 \implies x^\mu = 0$. We have $\omega_g = \lambda$ and $T_g = D$:

$$
D\Phi_a(0) = \tilde{\Delta}\Phi_a(0) \tag{7.22}
$$

corresponding commutator (by operating it on eigenstate of dilatation)

$$
\begin{aligned}
D\left|\Delta\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle + \Phi_\Delta(0) D \left|0\right\rangle \\
\tilde{\Delta}\left|\Delta\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle + 0 \\
\tilde{\Delta}\Phi_\Delta(0)\left|0\right\rangle &= [D, \Phi_\Delta(0)]\left|0\right\rangle
\end{aligned}
$$

Applying the same procedure, we consider:

$$
\begin{aligned}
e^{ix^\beta P_\beta} D e^{-ix^\sigma P_\sigma} &= D + [D, -ix^\beta P_\beta] + \frac{1}{2!}[[D, -ix^\beta P_\beta], -ix^\beta P_\beta] + \ldots \\
&= D - ix^\alpha(iP_\alpha) \\
&= D + x^\alpha P_\alpha \\
&= D - ix^\alpha \partial_\alpha
\end{aligned} \tag{7.23}
$$

for the sake of consistency at $x' = 0$

$$
= \tilde{\Delta} - ix^\alpha \partial_\alpha
$$

Now, we consider

$$
D\Phi_a(x) = (\tilde{\Delta} - ix^\alpha \partial_\alpha)\Phi_a(x)
$$

redefining $\tilde{\Delta} \equiv -i\Delta$, we get

$$
D\Phi_a(x) = -i(\Delta + x^\alpha \partial_\alpha)\Phi_a(x)
$$

Similarly,[8]

$$
\begin{aligned}
[D, \Phi_a(x)] &= De^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0)e^{-ix^\sigma P_\sigma} D \\
&= e^{ix^\lambda P_\lambda} \underbrace{e^{ix^\alpha P_\alpha} D e^{-ix^\beta P_\beta}}_{=D + x^\alpha P_\alpha} \Phi_a(0) e^{-ix^\sigma P_\sigma} - e^{ix^\beta P_\beta}\Phi_a(0) \underbrace{e^{-ix^\sigma P_\sigma} D e^{-ix^\alpha P_\alpha}}_{=D + x^\alpha P_\alpha} e^{-ix^\lambda P_\lambda} \\
&= e^{ix^\beta P_\beta}[D + x^\alpha P_\alpha, \Phi_a(0)]e^{-ix^\sigma P_\sigma} \\
&= e^{ix^\beta P_\beta}\underbrace{[D, \Phi_a(0)]}_{\tilde{\Delta}\Phi_a(0)} e^{-ix^\sigma P_\sigma} + e^{ix^\beta P_\beta}\underbrace{[x^\alpha P_\alpha, \Phi_a(0)]}_{=x^\alpha[P_\alpha, \Phi_a(0)]} e^{-ix^\sigma P_\sigma} \\
&= \tilde{\Delta}\Phi_a(x) - ix \cdot \partial\Phi_a(x) \\
&= -i(\Delta + x \cdot \partial)\Phi_a(x)
\end{aligned}
$$

Finite Dilatation[9], we consider

$$
x' = e^\lambda x = e^{i\lambda D} x = \left(1 + i\frac{\lambda}{N}\overset{\overset{Dx^\mu = -ix\cdot\partial x^\mu}{\downarrow}}{D}\right)\ldots\left(1 + i\frac{\lambda}{N}D\right)x
$$

[8]from pg 31 of [21], and x is not an operator here but a number

[9]look up *Lectures Notes For An Introduction to Conformal Field Theory A Course Given By Dr. Tobias Osborne*, pg 19

then at origin, the field transforms (active transformation) as:

$$\begin{aligned}
\Phi'_a(0) &= \left(1 + i\frac{\lambda}{N}D\right)\dots\left(1 + i\frac{\lambda}{N}D\right)\Phi_a(0) \\
&= \left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) && \text{(using } D\Phi(0) = \tilde{\Delta}\Phi) \\
&= e^{i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{-\lambda\Delta_a}\Phi_a(0)
\end{aligned}$$

In passive transformation

$$\begin{aligned}
\Phi'_a(0) &= \left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right)\dots\left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right)\Phi_a(0) \\
&= e^{-i\lambda\tilde{\Delta}_a}\Phi_a(0) \\
&= e^{\lambda\Delta}\Phi_a(0)
\end{aligned}$$

For arbitrary point (ignoring the change in argument of field and thus generator c_g):

$$\begin{aligned}
\Phi'_a(x') &= \pi_{ab}(e^{i\lambda D})\Phi_b(x) \\
&= [e^{i\lambda\tilde{\Delta}}]_{ab}\Phi_b(x) \\
\Phi'_a(e^{\lambda}x) &= [e^{-\lambda\Delta}]_{ab}\Phi_b(x) = e^{-\lambda\Delta}\Phi_a(x)
\end{aligned}$$

For SCT, $x'^{\mu} = 0 \implies x^{\mu} = 0$. Hence, we will consider the same equations, but in this context:

$$K_{\mu}\Phi_a(0) = \kappa_{\mu}\Phi_a(0)$$

Again, applying the same procedure,

$$\begin{aligned}
e^{ix^{\beta}P_{\beta}}K_{\mu}e^{-ix^{\sigma}P_{\sigma}} &= K_{\mu} + [K_{\mu}, -ix^{\beta}P_{\beta}] + \frac{1}{2!}[[K_{\mu}, -ix^{\beta}P_{\beta}], -ix^{\beta}P_{\beta}] + \dots \\
&= K_{\mu} - ix^{\beta}[K_{\mu}, P_{\beta}] + \frac{1}{2}[-ix^{\beta}[K_{\mu}, P_{\beta}], -ix^{\beta}P_{\beta}] + \dots \\
&= K_{\mu} + 2x^{\beta}(g_{\mu\beta}D - L_{\mu\beta}) + \frac{1}{2}[2x^{\beta}(g_{\mu\beta}D - L_{\mu\beta}), -ix^{\alpha}P_{\alpha}] \\
&= K_{\mu} + 2x_{\mu}D - 2x^{\beta}L_{\mu\beta} - ix_{\mu}x^{\alpha}[D, P_{\alpha}] + ix^{\beta}x^{\alpha}\underbrace{[L_{\mu\beta}, P_{\alpha}]}_{-i(g_{\alpha\mu}P_{\beta} - g_{\alpha\beta}P_{\mu})} \\
&= K_{\mu} + 2x_{\mu}D - 2x^{\beta}L_{\mu\beta} + x_{\mu}x^{\alpha}P_{\alpha} + x_{\mu}x^{\beta}P_{\beta} - x_{\alpha}x^{\alpha}P_{\mu} \\
&= K_{\mu} + 2x_{\mu}D - 2x^{\beta}L_{\mu\beta} + 2x_{\mu}x^{\alpha}P_{\alpha} - x_{\alpha}x^{\alpha}P_{\mu}
\end{aligned}$$

From the generator of dilatation and SCT, we have

$$[D, K_{\mu}] = -iK_{\mu} \qquad \text{at } x'^{\mu} = 0 \implies \qquad [\tilde{\Delta}, \kappa_{\mu}] = -i\kappa_{\mu}$$

and

$$[D, L_{\mu\nu}] = 0 \qquad \text{at } x'^{\mu} = 0 \implies \qquad [\tilde{\Delta}, S_{\mu\nu}] = 0$$

For primary fields:

$$K_{\mu}\Phi_a(0) = 0$$

Since, for primary field $\tilde{\Delta}$ commutes with all other operators which belong to the stability subgroup. By Schur's lemma $\tilde{\Delta} \propto I$, where I is an identity operator. The SCT and momentum generator acts as ladder operator for Dilatation.

$$\begin{aligned}
[D, [P_{\mu}, \Phi(0)]] &= [P_{\mu}, [D, \Phi(0)]] + [[D, P_{\mu}], \Phi(0)] = -i(\Delta + 1)[P_{\mu}, \Phi(0)] \\
[D, [K_{\mu}, \Phi(0)]] &= [K_{\mu}, [D, \Phi(0)]] + [[D, K_{\mu}], \Phi(0)] = -i(\Delta - 1)[K_{\mu}, \Phi(0)]
\end{aligned}$$

Finite Conformal Transformation of Fields

We begin by noting that *translation* and *rotation* do not introduce any new thing that we hadn't encountered in QFT, it is only the dilatation and SCT which does. Upon exponentiating the infinitesimal dilatation:

$$\begin{aligned}\Phi'(x') &= e^{-i\omega_g\left[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}\right]}\Phi(x') \\ &= e^{-i\omega_g T_g} e^{-i\omega_g c_g x'\cdot\frac{\partial}{\partial x'}}\Phi(x') \\ &= e^{-i\omega_g T_g}\Phi(e^{-i\omega_g c_g}x')\end{aligned}$$

The source of confusion here is that we have defined our generator in such a way that it also includes the translation. Due to this, we have x' instead of x. If we break it down further, we see that it resembles the old expression we find in other textbooks.

$$\begin{aligned}\Phi_a'(x') = U(\Lambda)\Phi_a(x')U^{-1}(\Lambda) &= e^{-i\omega_g T_g}\Phi_a e^{i\omega_g T_g} \\ &= e^{-i\omega_g[T_g,\ ]}\Phi_a(x')\end{aligned}$$

For translation

$$\Phi(x) = e^{ix^\lambda P_\lambda}\Phi(0)e^{-ix^\alpha P_\alpha} = e^{x\partial}\Phi(0)$$

Or,

$$\begin{aligned}\Phi'(x') &= \Phi(x) \\ &= \Phi(x'-a) \\ &= e^{-a\frac{\partial}{\partial x'}}\Phi(x') \\ &= e^{-iaP}\Phi(x')\end{aligned}$$

For dilatation ($x' = e^\lambda x$)

$$\begin{aligned}\Phi_a'(x') &= e^{-i\lambda D}\Phi_a(x') \\ &= e^{-\lambda(\Delta + x'\cdot\partial)}\Phi_a(x') \\ &= e^{-\lambda\Delta}\underbrace{e^{-\lambda x\cdot\partial}\Phi_a(x')}_{\Phi_a[e^{-\lambda}x']} \\ &= e^{-\lambda\Delta}\Phi_a(x)\end{aligned}$$

The last part could be understood as:

$$\begin{aligned}\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right] &= e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a(x) \\ \Phi_a\Bigg[\underbrace{\left(1-\frac{\lambda}{N}\right)\dots\left(1-\frac{\lambda}{N}\right)}_{N\text{ terms}}x\Bigg] &= e^{-\frac{\lambda}{N}x\cdot\partial}\Phi_a\Bigg[\underbrace{\left(1-\frac{\lambda}{N}\right)\dots\left(1-\frac{\lambda}{N}\right)}_{N-1\text{ terms}}x\Bigg] \\ \Phi_a(e^{-\lambda}x) &= e^{-\lambda x\cdot\partial}\Phi_a(x)\end{aligned}$$

or, alternatively

$$\begin{aligned}e^{-\lambda x\cdot\partial}\Phi_a(x) &= \left(1-\frac{\lambda}{N}x\cdot\partial\right)^N\Phi_a(x) \\ &= \left(1-\frac{\lambda}{N}x\cdot\partial\right)\dots\underbrace{\left(1-\frac{\lambda}{N}x\cdot\partial\right)\Phi_a(x)}_{\Phi_a\left[\left(1-\frac{\lambda}{N}\right)x\right]} \\ &= \Phi_a\left[\left(1-\frac{\lambda}{N}\right)^N x\right] \\ &= \Phi_a(e^{-\lambda}x)\end{aligned}$$

For SCT (on primary fields)

$$\begin{aligned}\Phi'(x') &= e^{-i\vec{b}\cdot\vec{K}}\Phi(x) \\ &= \Phi(x)\end{aligned}$$

Appendix: Conformal Ward Identity

We will now derive the conformal ward identity, which are crucial for bootstrap philosophy as well as we used it for deriving the Three Point Function involving conformally coupled scalar. The metric corresponding to de Sitter spacetime in flat slicing is given by:

$$ds^2 = \frac{-d\eta^2 + d\mathbf{x}^2}{\eta^2 H^2}$$

The generator of dilatation looks like:

$$-iD = -x \cdot \partial$$

The generator of SCT looks like:

$$\begin{aligned} -iK_\mu &= -[2x_\mu x \cdot \partial - x^2 \partial_\mu] \\ &= -2x_\mu \eta \partial_\eta - 2x_\mu \vec{x} \cdot \partial_{\vec{x}} - (\eta^2 - |\vec{x}|^2)\partial_\mu \end{aligned}$$

The infinitesimal special conformal transformation is given as:

$$\begin{aligned} x'^\mu &= x^\mu + ib^\nu K_\nu x^\mu \\ &\implies \underbrace{b^\nu K_\nu}_{b \cdot K} x^\mu = 0 \end{aligned}$$

Moving forward we will consider the SCT parametrized by $b^\mu = (0, \mathbf{b})$, where $\mathbf{b}$ is any arbitrary constant 3-vector.[10] Then, the infinitesimal conformal transformation which respects the de Sitter isometry will look like

$$\begin{aligned} \text{dilation:} \quad & \eta \to \eta(1+\lambda), \qquad && \mathbf{x} \to \mathbf{x}(1+\lambda), \\ \text{SCT:} \quad & \eta \to \eta(1 - 2\mathbf{b}\cdot\mathbf{x}), \qquad && \mathbf{x} \to \mathbf{x} - 2(\mathbf{b}\cdot\mathbf{x})\mathbf{x} + (\mathbf{x}^2 - \eta^2)\mathbf{b}, \end{aligned}$$

We consider the following quantity:

$$\begin{aligned} e^{i\lambda D} \langle \ldots \rangle = \langle \ldots \rangle &\implies D \langle \ldots \rangle = 0 \\ e^{i\vec{b}\cdot\vec{K}} \langle \ldots \rangle = \langle \ldots \rangle &\implies \vec{b} \cdot \vec{K} \langle \ldots \rangle \equiv \mathbf{b} \cdot \mathbf{K} \ \langle \ldots \rangle \ = 0 \end{aligned}$$

Since, at late time ($\eta \to 0$) we will be decomposing our fields like (refer (4.7)):

$$\Phi = \sum_{\{\Delta\}} \eta^\Delta O_\Delta(\vec{x})$$

The generators act on the boundary operator O_Δ like:

$$\begin{aligned} D\Phi &= -x \cdot \partial \Phi \\ &= -x \cdot \partial \sum_{\{\Delta\}} \eta^\Delta O_\Delta \\ &= \sum_{\{\Delta\}} (-\eta \partial_\eta - \vec{x} \partial_{\vec{x}}) \eta^\Delta O_\Delta \\ &= \sum_{\{\Delta\}} \eta^\Delta \underbrace{(-\Delta - \vec{x}\partial_{\vec{x}})}_{D} O_\Delta \\ &= D \sum_{\{\Delta\}} \eta^\Delta O_\Delta \end{aligned}$$

[10] field theory in cosmology by Enrico Pajer pg 59

$$
\begin{aligned}
b \cdot K\Phi &= b^\mu[-2x_\mu \eta \partial_\eta - 2x_\mu \vec{x} \cdot \partial_{\vec{x}} - (\eta^2 - |\vec{x}|^2)\partial_\mu]\Phi \\
&= [-2(\vec{b} \cdot \vec{x})\eta\partial_\eta - 2(\vec{b} \cdot \vec{x})\vec{x} \cdot \partial_{\vec{x}} - (\eta^2 - |\vec{x}|^2)\vec{b} \cdot \partial_{\vec{x}}]\Phi \\
&= \sum_{\{\Delta\}} \underbrace{[-2(\vec{b} \cdot \vec{x})\Delta - 2(\vec{b} \cdot \vec{x})\vec{x} \cdot \partial_{\vec{x}} - (\eta^2 - |\vec{x}|^2)\vec{b} \cdot \partial_{\vec{x}}]}_{b \cdot K} \eta^\Delta O_\Delta
\end{aligned}
$$

in the limit $\eta \to 0$

$$
= [-2(\vec{b} \cdot \vec{x})\Delta - 2(\vec{b} \cdot \vec{x})\vec{x} \cdot \partial_{\vec{x}} + |\vec{x}|^2\vec{b} \cdot \partial_{\vec{x}}]\Phi
$$

In momentum space, the above operators take the following form:

$$
\begin{aligned}
D &: (3 - \eta\partial_\eta) + k^i \partial_{k_i}, \\
\mathbf{b} \cdot \mathbf{K} &: (3 - \eta\partial_\eta)\, 2b^i \partial_{k_i} - \mathbf{b} \cdot \mathbf{k} \partial_{k_i}\partial_{k_i} + 2k^i \partial_{k_i} b^j \partial_{k_j}.
\end{aligned}
$$

or, simply

$$
\begin{aligned}
D &: (3 - \Delta) + k^i \partial_{k_i}, \\
\mathbf{b} \cdot \mathbf{K} &: (3 - \Delta)\, 2b^i \partial_{k_i} - \mathbf{b} \cdot \mathbf{k} \partial_{k_i}\partial_{k_i} + 2k^i \partial_{k_i} b^j \partial_{k_j}.
\end{aligned}
$$

There are two ways to derive this expression, we will discuss both of them. The first is based on using Fourier Transform:

$$
f(\vec{x}) = \int d^3x e^{i\vec{x}\cdot\vec{k}} f(\vec{k})
$$

In our case

$$
\begin{aligned}
D \underbrace{\int d^3k e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k})}_{O_\Delta(\vec{x})} &= -\left(\Delta + x^j \frac{\partial}{\partial x^j}\right) \int d^3k e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k}) \\
&= \int d^3k \left[-\Delta - x^j \frac{\partial}{\partial x^j}\right] e^{i\vec{x}\cdot\vec{k}} O_\Delta(\vec{k}) \\
&= \int d^3k [-\Delta e^{i\vec{x}\cdot\vec{k}} - x^j e^{i\vec{x}\cdot\vec{k}}(ik_j)] O_\Delta(\vec{k})
\end{aligned}
$$

we have to get rid of x^j, so we consider:

$$
\begin{aligned}
&= \int d^3k \left[-\Delta e^{i\vec{x}\cdot\vec{k}} - \left(-i\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right) ik_j\right] O_\Delta(\vec{k}) \\
&= \int d^3k \left[-\Delta e^{i\vec{x}\cdot\vec{k}} - \left(\frac{\partial e^{i\vec{x}\cdot\vec{k}}}{\partial k_j}\right) k_j\right] O_\Delta(\vec{k})
\end{aligned}
$$

integrating by parts the second term

$$
\begin{aligned}
&= \int d^3k e^{i\vec{x}\cdot\vec{k}} \left[-\Delta + \left(\frac{\partial}{\partial k_j}\right) k_j\right] O_\Delta(\vec{p}) \\
&= \int d^3k e^{i\vec{x}\cdot\vec{k}} \left(-\Delta + \cancelto{3}{\frac{\partial k_j}{\partial k_j}} + k_j \frac{\partial}{\partial k_j}\right) O_\Delta(k)).
\end{aligned}
$$

Thus the action of the dilatation generator in momentum space is

$$
DO_\Delta(\vec{k}) = \left(3 - \Delta + k^j \frac{\partial}{\partial k^j}\right) O_\Delta(\vec{k})
$$

The second way is to replace the following in the expression in coordinate space (it can be seen operating the corresponding derivative operator on $e^{ix^\mu k_\mu}$ and the corresponding integration by parts).

$$
\begin{aligned}
x_\mu &\to -i\frac{\partial}{\partial k^\mu} \\
\frac{\partial}{\partial x_\mu} &\to -ik^\mu
\end{aligned}
$$

Substituting in

$$DO_\Delta(\vec{x}) = -\left(\Delta + x^j \frac{\partial}{\partial x^j}\right) O_\Delta(\vec{x})$$

we get

$$DO_\Delta(\vec{k}) = -\left[\Delta - i\frac{\partial}{\partial k^j}\left(-ik^j\right)\right] O_\Delta(\vec{k}) = -\left(\Delta - 3 - k^j\frac{\partial}{\partial k^j}\right) O_\Delta(\vec{k})$$

Conformal Ward Identity in momentum space

We start by assuming that the generator of conformal transformation annihilates the correlation function.

$$\begin{aligned} D\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\right\rangle &= 0 \\ e^{ix_n\cdot P} D\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\right\rangle e^{-ix_n\cdot P} &= 0 \\ \underbrace{e^{ix_n\cdot P} D e^{-ix_n\cdot P}}_{D+\vec{x}_n\cdot\partial_{\vec{x}_n}} \underbrace{e^{ix_n\cdot P}\left\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\right\rangle e^{-ix_n\cdot P}}_{\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\rangle} &= 0 \end{aligned}$$

now, in Fourier space:

$$\begin{aligned} \left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\right\rangle &= \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n-1} ik_j\cdot(x_j-x_n)+0} \\ &\quad\times \delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\ &= \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n-1} ik_j\cdot x_j - ix_n\cdot(\sum_{j=1}^{n-1}k_j)} \\ &\quad\times \delta^d\left(k_n + \sum_j^{n-1} k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\ &\equiv \int d^dk_1\dots d^dk_n e^{\sum_{j=1}^{n} ik_j\cdot x_j} \\ &\quad\times \delta^d\left(\sum_j^n k_j\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \end{aligned}$$

From first and third equality, we observe that the replacement we need to make are:

$$(x_j - x_n)_\mu \to -i\frac{\partial}{\partial k_j^\mu}$$
$$\frac{\partial}{\partial x_\mu} \to -ik^\mu$$

Thus,

$$\begin{aligned} (\underbrace{D + \vec{x}_n\cdot\partial_{\vec{x}_n}}_{-(\sum_{j=1}^n \Delta_j + x_j\cdot\partial_{x_j}) + x_n\partial_{x_n}})\left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\right\rangle &= 0 \\ -\left[\sum_j^n \Delta_j + \sum_j^{n-1}(x_j - x_n)\cdot\partial_{x_j}\right]\left\langle O_1(\vec{x}_1-\vec{x}_n)O_2(\vec{x}_2-\vec{x}_n)\dots O_n(0)\right\rangle &= 0 \\ -\underbrace{\left[\sum_j^n \Delta_j + \sum_j^{n-1}\left(-i\frac{\partial}{\partial k_j^\mu}\right)\cdot(-ik_j^\mu)\right]}_{-[\Delta-(n-1)d-\sum_{j=1}^{n-1}k_j\cdot\partial_{k_j}]}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle\delta^d\left(\sum_j^n k_j\right) &= 0 \end{aligned}$$

The alternate way to do the same is by explicitly doing it. We then will have to use

$$\int dx f(x)\partial_x\delta(x-a) = -\int dx\partial_x f(x)\delta(x-a) = -\partial_x f(a)$$

Consider the following integral:

$$
\begin{aligned}
I_{\alpha\beta} &= \int d^d k \left[\frac{\partial}{\partial x^\alpha}\delta^d(k^\mu)\right] k_\beta \\
&= -\int d^d k \delta^d(k^\mu)\underbrace{\frac{\partial k_\beta}{\partial x^\alpha}}_{g_{\alpha\beta}} = -g_{\alpha\beta}
\end{aligned}
\tag{7.24}
$$

However, we also know that:

$$
\begin{aligned}
\int d^d k \delta^d(k^\mu) &= 1 \\
\int d^d k \delta^d(k^\mu)\frac{k^2}{k^2} &= 1 \\
\int d^d k \delta^d(k^\mu)\frac{g^{\alpha\beta}k_\alpha k_\beta}{k^2} &= 1 \\
\int d^d k \frac{\delta^d(k^\mu)}{k^2} k_\alpha k_\beta &= \frac{1}{d} g_{\alpha\beta}
\end{aligned}
\tag{7.25}
$$

from (7.24) and (7.25), we get

$$
\frac{\partial}{\partial k^\alpha}\delta^d(k^\mu) = -\frac{d}{k^2} k_\alpha \delta^d(k^\mu)
$$

using the above, we derive:

$$
k^\alpha \frac{\partial}{\partial k^\alpha}\delta^3(\vec{k}) = -\frac{3}{k^2} k^\alpha k_\alpha \delta^3(\vec{k}) = -3\delta^3(\vec{k})
$$

Then, we have:

$$
\begin{aligned}
-\sum_{j=1}^{n}&\left(\Delta_j + x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right)\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle \\
&= -\sum_{j=1}^{3}\left(\Delta_j + x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right) \\
&\qquad \int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{ik_l x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\
&= -\int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{ik_l x_l} \\
&\qquad \left(\sum_{j=1}^{n}\Delta_j - \underbrace{\sum_{j=1}^{n} d}_{nd} - \sum_{j=1}^{n} k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle + \delta'_{\text{term}}
\end{aligned}
$$

where

$$
\delta'_{\text{term}} = \sum_{j=1}^{n}\int \prod_{l=1}^{n} d^d k_l \left[k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\delta^d(\textstyle\sum_{i=1}^n \vec{k}_i)\right] e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle
$$

defining $P = \sum_{i=1}^{n} \vec{k}_i$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\int \prod_{l=1}^{n} d^d k_l \left[k_j^\alpha \frac{\partial}{\partial P^\alpha}\delta^d(P)\right] e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\
&= \int \prod_{l=1}^{n} d^d k_l \left[P^\alpha \frac{\partial}{\partial P^\alpha}\delta^d(P)\right] e^{ik_l\cdot x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \\
&= -d\int \prod_{l=1}^{n} d^d k_l \delta^d(\textstyle\sum_{i=1}^n \vec{k}_i) e^{ik_l x_l}\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle
\end{aligned}
$$

Thus, finally

$$-\sum_{j=1}^{n}\left(x_j^\alpha\frac{\partial}{\partial x_j^\alpha}+\Delta_j\right)\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\dots O_n(\vec{x}_n)\rangle = -\int\prod_{l=1}^{n}d^dk_l\delta^d(\textstyle\sum_{i=1}^n k_i)e^{ik_lx_l}$$
$$\left(\sum_{j=1}^{n}\Delta_j-(n-1)d-\sum_{j=1}^{n}k_j^\alpha\frac{\partial}{\partial k_j^\alpha}\right)\left\langle O_1(\vec{k}_1)O_2(\vec{k}_2)\dots O_n(\vec{k}_n)\right\rangle \quad (7.26)$$

Before we proceed, there is another identity we'd like to derive which would be very helpful.

$$\begin{aligned}\partial_\alpha\partial_\beta\delta^d(k^\mu) &= \partial_\alpha\left[-\frac{d}{k^2}\delta^d(k^\mu)k_\beta\right]\\ &= -d\left(\frac{\partial}{\partial k^\alpha}k^{-2}\right)\delta^d(k^\mu)k_\beta-\frac{d}{k^2}[\partial_\alpha\delta^d(k)]k_\beta-\frac{d}{k^2}\delta^d(k^\mu)\partial_\alpha k_\beta\\ &= \frac{d(d+2)}{k^4}\delta^d(k^\mu)k_\alpha k_\beta-\frac{d}{k^2}\delta^d(k^\mu)g_{\alpha\beta}\end{aligned}$$

We will now discuss the same for SCT. We perform the same substitution

$$\begin{aligned}-iK &= -2x_\mu\Delta-2x_\mu\underbrace{\vec{x}\cdot\partial_{\vec{x}}}_{-i\frac{\partial}{\partial k^\alpha}(-ik^\alpha)}+|\vec{x}|^2\partial_\mu\\ &= 2i\Delta\frac{\partial}{\partial k^\mu}-2i\frac{\partial^2}{\partial k^\mu\partial k^\alpha}k^\alpha+i\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}k_\mu\\ &= 2i\Delta\frac{\partial}{\partial k^\mu}-\underbrace{2i\frac{\partial}{\partial k^\mu}\left(d+k^\alpha\frac{\partial}{\partial k^\alpha}\right)}_{2i\left(d\frac{\partial}{\partial k^\mu}+\frac{\partial}{\partial k^\mu}+k^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha}\right)}+\underbrace{i\frac{\partial}{\partial k^\alpha}\left(\delta_\mu^\alpha+k_\mu\frac{\partial}{\partial k_\alpha}\right)}_{2i\frac{\partial}{\partial k^\mu}+ik_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}}\\ &= 2i\Delta\frac{\partial}{\partial k^\mu}-2id\frac{\partial}{\partial k^\mu}-2ik^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha}+ik_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}\\ &= i\left[2(\Delta-d)\frac{\partial}{\partial k^\mu}-2k^\alpha\frac{\partial^2}{\partial k^\mu\partial k^\alpha}+k_\mu\frac{\partial^2}{\partial k^\alpha\partial k_\alpha}\right]\end{aligned}$$

Then,

$$\begin{aligned}K^\mu\delta^d(\textstyle\sum_{i=1}^n p_i^k) &= \sum_{j=1}^{n}\left[k_j^\mu\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\alpha}}-2k_j^\alpha\frac{\partial^2}{\partial k_j^\alpha\partial k_{j\mu}}+2(\Delta_j-d)\frac{\partial}{\partial k_{j\mu}}\right]\delta^d(\underbrace{\textstyle\sum_{i=1}^n k_i^\mu}_{P^\mu=\sum_{i=1}^n k_i^\mu})\\ &= \sum_{j=1}^{n}\left[k_j^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha}-2k_j^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu}+2(\Delta_j-d)\frac{\partial}{\partial P_\mu}\right]\delta^d(P)\\ &= \left[\left(\sum_{j=1}^{n}k_j^\mu\right)\frac{\partial^2}{\partial P^\alpha\partial P_\alpha}-2\left(\sum_{j=1}^{n}k_j^\alpha\right)\frac{\partial^2}{\partial P^\alpha\partial P_\mu}+2\sum_{j=1}^{n}(\Delta_j-d)\frac{\partial}{\partial P_\mu}\right]\delta^d(P)\\ &= \left[P^\mu\frac{\partial^2}{\partial P^\alpha\partial P_\alpha}-2P^\alpha\frac{\partial^2}{\partial P^\alpha\partial P_\mu}+2(\Delta-nd)\frac{\partial}{\partial P_\mu}\right]\delta^d(P)\\ &= \frac{2d}{P^2}\delta^d(P)P^\mu-2\frac{d^2+d}{P^2}\delta^d(P)P^\mu+\frac{2d(\Delta-nd)}{P^2}\delta^d(P)P^\mu\\ &= 2d\left[nd-d-\Delta\right]\frac{\delta^d(P)P^\mu}{P^2}\\ &= -2[(n-1)d-\Delta]\frac{\partial\delta^d(P)}{\partial P^\mu}\end{aligned}$$

In the fourth line, we used $\sum_{j=1}^n\Delta_j=\Delta$. Now, when K operates on the correlation function, it produces three kinds of terms,

- All operators in K acting purely on $\langle O_1(p_1)\dots O_n(p_n)\rangle$
- All operators in K acting purely on $\delta^d(\sum_{i=1}^n p_i)$

- Operators acting on both $\langle O_1(p_1)\dots O_n(p_n)\rangle$ and $\delta^d(\sum_{i=1}^n p_i)$

We will consider the action of

$$k_j^\mu \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\alpha}} - 2k_j^\alpha \frac{\partial^2}{\partial k_j^\alpha \partial k_{j\mu}}$$

then, they will operate like:

$$2\frac{\partial \delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\alpha} \underbrace{\left[k_j^\mu \frac{\partial}{\partial k_{j\alpha}} - k_j^\alpha \frac{\partial}{\partial k_{j\mu}}\right]}_{iL_{\mu\alpha}} \left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$
$$- 2\frac{\partial \delta^d(\sum_{i=1}^n k_i^\mu)}{\partial k_j^\mu} k_j^\alpha \frac{\partial}{\partial k_j^\alpha} \left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

First part vanishes due to rotational invariance. Therefore the extra terms would be:

$$\delta'_{\text{terms}} = -2\frac{\partial \delta^d(P)}{\partial P}\left[(n-1)d - \Delta + \sum_{j=1}^n k_j^\alpha \frac{\partial}{\partial k_j^\alpha}\right]\left\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\right\rangle$$

The above vanishes from (7.26). Therefore the SCT ward identity is given as[22]:

$$K_\mu \langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle$$
$$= -\left[2(\Delta - d)\frac{\partial}{\partial k^\mu} - 2k^\alpha \frac{\partial^2}{\partial k^\mu \partial k^\alpha} + k_\mu \frac{\partial^2}{\partial k^\alpha \partial k_\alpha}\right]\langle O_1(\vec{k}_1)\dots O_n(\vec{k}_n)\rangle \quad (7.27)$$

Appendix: Embedding Space Formalism

Let us consider the following embedding space coordinates.

$$X^0, \underbrace{X^1, X^2, \ldots, X^d}_{X^\mu}, X^{d+1}$$

We define the following null coordinates,

$$X^M = (X^+, X^-, X^\mu),$$

where

$$\left.\begin{aligned} X^+ &= X^0 + X^{d+1} \\ X^- &= X^0 - X^{d+1} \end{aligned}\right\} X^0 = \frac{X^+ + X^-}{2}; \quad X^{d+1} = \frac{X^+ - X^-}{2}$$

with the mostly plus metric in $\mathbb{R}^{d+1,1}$, reads as

$$ds^2 = \sum_{\mu=1}^{d} (dX^\mu)^2 - dX^+ dX^-$$

with the metric given as:

$$\eta_{MN} = \begin{pmatrix} 0 & -1/2 & 0 & 0 & \cdots \\ -1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 0 & 1 & \\ & & & & \ddots \end{pmatrix}$$

Needless to say it is extremely simpler to construct Lorentz covariant expressions than conformal covariant ones. Therefore the problem is: once we have constructed Lorentz covariant expressions in $d+2$ dimensions how do we descend to d dimensions without breaking the covariance? This can be done as follows:

We first note that null light cone $X^2 = 0$ is Lorentz Invariant in embedding space. Therefore, we will consider a null cone in the Minkowski space $\mathbb{R}^{d+1,1}$, i.e. the space of null rays passing through origin defined via:

$$\begin{aligned} X^2 &= -(X^0)^2 + (X^1)^2 + \ldots + (X^{d+1})^2 \\ &= -X^+X^- + \underbrace{(X^1)^2 + \ldots + (X^d)^2}_{\sum_{\mu=1}^{d}(X^\mu)^2} \\ &= 0 \end{aligned}$$

We can use this constraint to remove one of the coordinates, say X^- out of $d+2$ coordinates in embedding space. Next, we will think of the embedding space as a fiber bundle over the $d-$dimensional base space (the physical space where the CFT is defined). The fibers in this fiber bundle are the null lines in the $(d+2)-$dimensional space that project down to points in the $d-$dimensional space. At each point in the base space, section (a map) is defined, which is a specific choice of null vector from each fiber. Points in the original $d-$dimensional space are represented by null vectors X^A subject to the identification $X^A \sim \lambda X^A$ for any non-zero λ. All null vectors isomorphic to each other up to dilation belong to the same fiber. This has an important consequence: Lorentz transformations map a point to another point outside the Euclidean section. To return to the Euclidean section, we need to use dilation, which is taken care of due to this identification. Therefore, in order to eliminate another one of the coordinate, say X^+, we consider an Euclidean section of the embedding space defined by:

$$X^+ = f(X^\mu) \equiv f(x^\mu)$$

This will help us identifying X^μ with the Euclidean space coordinates x^μ.

$$X^\mu \equiv x^\mu$$

This leads to definition of X^- based on null condition as:

$$X^- = \frac{\sum_{\mu=1}^{d}(X^\mu)^2}{X^+} = \frac{x^2}{f(X^\mu)}$$

The spacetime interval on this section is given as:

$$ds^2 = dx^2 - dX^+dX^-\Big|_{X^+=f(X^i),X^-=\frac{x^2}{X^+}}$$

This section satisfies two of the following conditions:

- section intersects each of the light rays at some point
- maps each point in d dimensional Euclidean space to a point on the null cone in Embedding space.

Let us now analyze how Lorentz Transformation acts on a generic section. The Lorentz transformation acting as rotation on the point X^A in the null-cone will move it to another point on the null cone outside the the section $X^B = \Lambda^B{}_A X^A$.

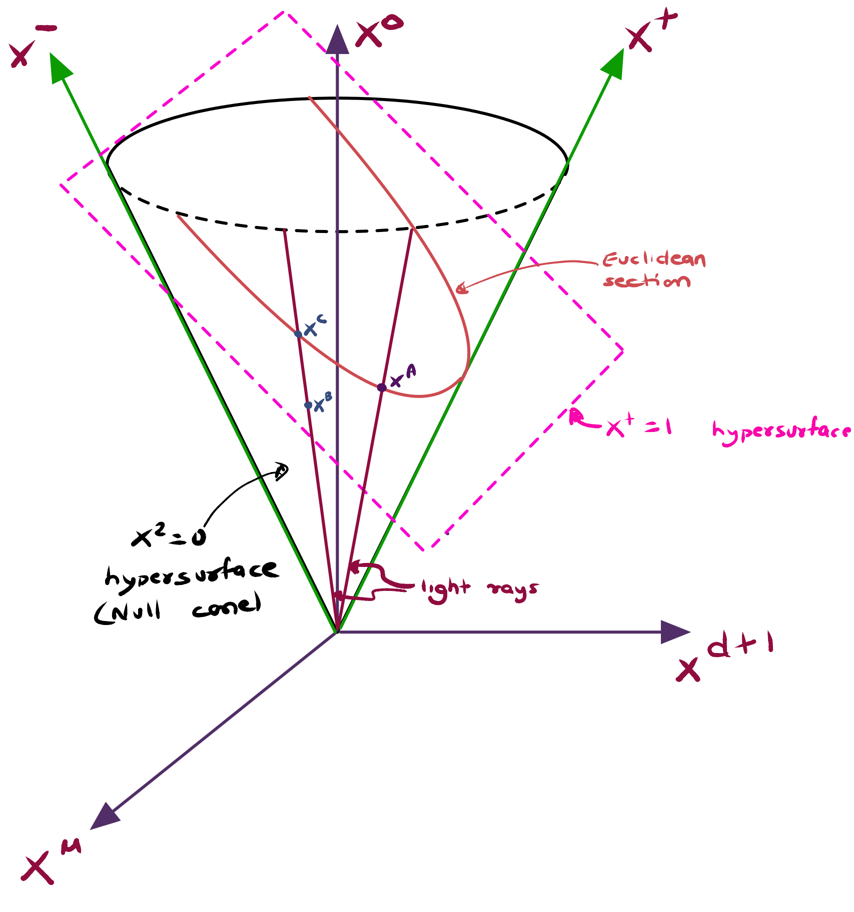


Figure 7.1: The hypersurface perpendicular to X^+ axis cutting at $X^+ = 1$ is shown as a plane and the null hypersurface is shown as the cone. The intersection of these two hypersurfaces describes the Euclidean Section.

However, suppose via some conformal transformation (dilatation) in d dimensional Euclidean Space, we can move X^B to X^C back into the section. In all these steps, the only thing that changes the metric is the rescaling to get back into the section.

$$\begin{aligned}
ds_B^2 &= dX^M dX_M \\
&= d(\lambda(X)X^M)d(\lambda(X)X_M) \\
&= [\lambda dX^M + X^M(\nabla\lambda\cdot dX)][\lambda dX_M + X_M(\nabla\lambda\cdot dX)] \\
&= \lambda^2 dX^M dX_M + 2\lambda \underbrace{dX^M X_M}_{=0}(\nabla\lambda\cdot dX) + \underbrace{X^M X_M}_{=0}(\nabla\lambda\cdot dX)^2 \\
&= \lambda^2 dX^M dX_M = \lambda^2 ds_C^2
\end{aligned}$$

where we used, $X^2 = 0$ and $X^\mu dX_\mu = 0$ for restricting it to null cone. Assuming the three conditions we used for simplification applies, the Lorentz Transformation in d+2-dimensional spacetime is equivalent to conformal transformation in $d-$dimensional spacetime if metric in $d-$dimensional space is Euclidean thus, dX_+ in ds^2 has to vanish. It gives us the condition for defining the Euclidean section as $X^+ = constant$ and thus, for the sake of simplicity, we take it as 1. Thus, we have two conditions which we can use to eliminate two extra degree of freedom.

In the embedding space formalism, choosing an Euclidean section corresponds to picking a specific way to embed the $d-$dimensional space in the $(d+2)-$dimensional space. We define the following map between d dimensional Euclidean Space with conformal symmetry to null cone in $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$

$$(X^+, X^-, X^\mu) \equiv \left(1, x^2, x^\mu\right)$$

Here, we note that choosing a constant value for X^+ would give us a section on the cone on which the induced metric is Euclidean.

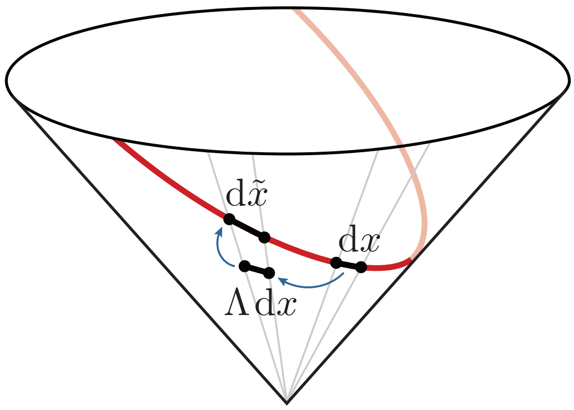


Figure 7.2: Upon Lorentz transformation, the points get mapped to different section however by utilizing the dilatation, we bring it back inside the original Euclidean section.

Tensors in Embedding Space

In this section, we will only concern ourselves with traceless and symmetric fields in $\mathbb{R}^d$ and leave the anti-symmetric tensors for future. Consider a symmetric and traceless tensor $O_{M_1\ldots M_S}$ defined on the cone $X^2 = 0$ in $\mathbb{R}^{d+1,1}$. Under the rescaling $X \to \lambda X$, the tensor transforms as

$$O_{M_1\ldots M_S}(\lambda X) = \lambda^{-\Delta} O_{M_1\ldots M_S}(X)$$

i.e. it is a homogeneous function of degree Δ. We expect $O_{M_1\ldots M_S}$ to get mapped to traceless and symmetric primary field in $\mathbb{R}^d$. Since, each index go from 0 to $d+1$, in $\mathbb{R}^{d+1,1}$ we find that, for $d+2-$dimensional fields other than scalar have 2 more degree of freedom per index than $d-$dimensional fields. In order to remove the extra degree of freedom, we consider the transversality condition.

$$X^{M_1} O_{M_1\ldots M_S} = 0$$

We define the physical field to be:

$$\phi_{\mu\nu\lambda\ldots}(x) = \frac{\partial X^{M_1}}{\partial x^\mu}\frac{\partial X^{M_2}}{\partial x^\nu}\frac{\partial X^{M_3}}{\partial x^\lambda}\ldots O_{M_1\ldots M_S}(X)\bigg|_{X=X(x)}$$

Note that, this definition implies a redundancy. Indeed, anything proportional to X^M gives zero since

$$X^2 = 0 \implies X_M \frac{\partial X^M}{\partial x^\mu} = 0$$

Therefore, $O_{M_1\ldots M_S}(X) \to O_{M_1\ldots M_S}(X) + X_{M_1} F_{M_2\ldots M_S}(X)$ gets mapped to the same physical field. This $SO(d+1,1)$ tensor is sometimes referred to as pure gauge in the literature. It is this gauge redundancy that reduces another degree of freedom per index by making it unphysical.

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